DIFFERENTIAL OPERATORS ON QUANTIZED FLAG MANIFOLDS AT ROOTS OF UNITY II

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Abstract. We formulate a Beilinson-Bernstein type derived equivalence for a quantized enveloping algebra at a root of 1 as a conjecture. It says that there exists a derived equivalence between the category of modules over a quantized enveloping algebra at a root of 1 with fixed regular Harish-Chandra central character and the category of certain twisted D-modules on the corresponding quantized flag manifold. We show that the proof is reduced to a statement about the (derived) global sections of the ring of differential operators on the quantized flag manifold. We also give a reformulation of the conjecture in terms of the (derived) induction functor.

0. Introduction

0.1. Let G be a connected, simply-connected simple algebraic group over $\mathbb{C}$, and let H be a maximal torus of G. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. Let Q and Λ be the root lattice and the weight lattice respectively. Let h_G be the Coxeter number of G. We fix an odd integer $\ell > h_G$, which is prime to the order of Λ/Q and prime to 3 if $\mathfrak{g}$ is of type G_2, and consider the De Concini-Kac type quantized enveloping algebra U_ζ at $q = \zeta = \exp(2\pi\sqrt{-1}/\ell)$.

In [20] we started the investigation of the corresponding quantized flag manifold $\mathcal{B}_\zeta$, which is a non-commutative scheme, and the category of D-modules on it. In view of a general philosophy saying that quantized objects at roots of 1 resemble ordinary objects in positive characteristics, it is natural to pursue analogue of the theory of D-modules on the ordinary flag manifolds in positive characteristics due to Bezrukavnikov-Mirković-Rumynin [4]. Along this line we have established in [20] certain Azumaya properties of the ring of differential operators on the quantized flag manifold. The aim of this paper is to investigate an analogue of another main point of [4] about the Beilinson-Bernstein type derived equivalence.

0.2. We denote by $\mathcal{D}_{\mathcal{B}_\zeta,1}$ the "sheaf of rings of differential operators" on the quantized flag manifold $\mathcal{B}_\zeta$. More generally, for each $t \in H$ we have

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its twisted analogue denoted by $\mathcal{D}_{\mathcal{B}_\zeta,t}$. It is obtained as the specialization $\mathcal{D}_{\mathcal{B}_\zeta}\otimes_{\mathbb{C}[H]}\mathbb{C}$ of the universally twisted sheaf $\mathcal{D}_{\mathcal{B}_\zeta}$ with respect to the ring homomorphism $\mathbb{C}[H]\to\mathbb{C}$ corresponding to $t\in H$.

Let $\mathcal{B}$ be the ordinary flag manifold for G. Then we have a Frobenius morphism $Fr:\mathcal{B}_\zeta\to\mathcal{B}$, which is a finite morphism from a noncommutative scheme to an ordinary scheme. Taking the direct images we obtain sheaves $Fr_*\mathcal{D}_{\mathcal{B}_\zeta}$, $Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$ ($t\in H$) of rings on $\mathcal{B}$ (in the ordinary sense). Denote by $\mathrm{Mod}_{coh}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ the category of coherent $Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. Let $Z_{Har}(U_\zeta)$ be the Harish-Chandra center of U_ζ, and let $\mathbb{C}_t$ be the corresponding one-dimensional $Z_{Har}(U_\zeta)$-module. Denote by $\mathrm{Mod}_f(U_\zeta\otimes_{Z_{Har}(U_\zeta)}\mathbb{C}_t)$ the category of finitely-generated $U_\zeta\otimes_{Z_{Har}(U_\zeta)}\mathbb{C}_t$-modules. Then we have a functor

$$R\Gamma(\mathcal{B},\bullet):D^b(\mathrm{Mod}_{coh}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}))\to D^b(\mathrm{Mod}_f(U_\zeta\otimes_{Z_{Har}(U_\zeta)}\mathbb{C}_t)) \tag{0.1}$$

between derived categories. It is natural in view of [4] to conjecture that (0.1) gives an equivalence if t is regular. By imitating the argument of [4] we can show that this is true if we have

$$R\Gamma(\mathcal{B},Fr_*\mathcal{D}_{\mathcal{B}_\zeta})\cong U_\zeta\otimes_{Z_{Har}(U_\zeta)}\mathbb{C}[\Lambda]. \tag{0.2}$$

However, we do not know how to prove (0.2) at present, and hence we can only state it as a conjecture. We have also a stronger conjecture

$$R\Gamma(\mathcal{B},Fr_*(\mathcal{D}_{\mathcal{B}_\zeta})_f)\cong U_{\zeta,f}\otimes_{Z_{Har}(U_\zeta)}\mathbb{C}[\Lambda], \tag{0.3}$$

which is the analogue of (0.2) regarding the adjoint finite parts $(\mathcal{D}_{\mathcal{B}_\zeta})_f$, $U_{\zeta,f}$ of $\mathcal{D}_{\mathcal{B}_\zeta}$, U_ζ, respectively. We will give a reformulation of (0.3) in terms of the induction functor (see Conjecture 5.2 below). It turns out that (0.3) is equivalent to an assertion in Backelin-Kremnizer [3] stated to be true except for finitely many ℓ's.

It is also an interesting problem to find a formulation which works even in the case when the parameter $t\in H$ is singular. In the case of Lie algebras in positive characteristics Bezrukavnikov-Mirković-Rumynin [5] have succeeded in giving a more general framework, which works even for singular parameter, using partial flag manifolds (quotients of G by parabolic subgroups). In their case the parameter space is $\mathfrak{h}^*$, and one can associate for each $h\in\mathfrak{h}^*$ a parabolic subgroup whose Levi subgroup is the centralizer of h; however, in our case the centralizer of $t\in H$ is not necessarily a Levi subgroup of a parabolic subgroup, and hence the method in [5] cannot be directly applied to our case.

0.3. The contents of this paper is as follows. In Section 1 we recall basic facts on quantized enveloping algebras at roots of 1 and the corresponding quantized flag manifolds. In Section 2 we investigate properties of the category of D-modules. In particular, we show that (0.2) implies (0.1) for regular t's, and (0.3) implies (0.2). In Section 3 and Section 4 we recall some known results on the representations of quantized enveloping algebras and the induction functor respectively.

Finally, in Section 5 we give a reformulation of (0.3) in terms of the induction functor.

1. Quantized flag manifold

1.1. **Quantized enveloping algebras.**

1.1.1. Let G be a connected simply-connected simple algebraic group over the complex number field $\mathbb{C}$. We fix Borel subgroups B^+ and B^- such that $H = B^+ \cap B^-$ is a maximal torus of G. Set $N^+ = [B^+, B^+]$ and $N^- = [B^-, B^-]$. We denote the Lie algebras of G, B^+, B^-, H, N^+, N^- by $\mathfrak{g}$, $\mathfrak{b}^+$, $\mathfrak{b}^-$, $\mathfrak{h}$, $\mathfrak{n}^+$, $\mathfrak{n}^-$ respectively. Let $\Delta \subset \mathfrak{h}^*$ be the root system of $(\mathfrak{g}, \mathfrak{h})$. We denote by $\Lambda \subset \mathfrak{h}^*$ and $Q \subset \mathfrak{h}^*$ the weight lattice and the root lattice respectively. For $\lambda \in \Lambda$ we denote by θ_λ the corresponding character of H. The coordinate algebra $\mathbb{C}[H]$ of H is naturally identified the group algebra $\mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e(\lambda)$ via the correspondence $\theta_\lambda \leftrightarrow e(\lambda)$ for $\lambda \in \Lambda$. We take a system of positive roots Δ^+ such that $\mathfrak{b}^+$ is the sum of weight spaces with weights in $\Delta^+ \cup \{0\}$. Let $\{\alpha_i\}_{i \in I}$ be the set of simple roots, and $\{\varpi_i\}_{i \in I}$ the corresponding set of fundamental weights. We denote by Λ^+ the set of dominant integral weights. We set $Q^+ = \bigoplus_{i \in I} \mathbb{Z}_{\geqq 0} \alpha_i$. Let $W \subset GL(\mathfrak{h}^*)$ be the Weyl group. For $i \in I$ we denote by $s_i \in W$ the corresponding simple reflection. We take a W-invariant symmetric bilinear form

$$(\; , \;) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$$

such that $(\alpha, \alpha) = 2$ for short roots α. For $\alpha \in \Delta$ we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. For $i \in I$ we fix $\overline{e}_i \in \mathfrak{g}_{\alpha_i}$, $\overline{f}_i \in \mathfrak{g}_{-\alpha_i}$ such that $[\overline{e}_i, \overline{f}_i] = \alpha_i^\vee$ under the identification $\mathfrak{h} = \mathfrak{h}^*$ induced by $(\; , \;)$.

1.1.2. For $n \in \mathbb{Z}_{\geqq 0}$ we set

$$[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}], \qquad [n]_t! = [n]_t [n-1]_t \cdots [2]_t [1]_t \in \mathbb{Z}[t, t^{-1}].$$

We denote by $U_{\mathbb{F}}$ the quantized enveloping algebra over $\mathbb{F} = \mathbb{Q}(q^{1/|\Lambda/Q|})$ associated to $\mathfrak{g}$. Namely, $U_{\mathbb{F}}$ is the associative algebra over $\mathbb{F}$ generated by elements

$$k_\lambda \quad (\lambda \in \Lambda), \qquad e_i, f_i \quad (i \in I)$$

satisfying the relations

$$
\begin{aligned}
&k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda+\mu} && (\lambda, \mu \in \Lambda),\\
&k_\lambda e_i k_\lambda^{-1} = q^{(\lambda,\alpha_i)} e_i, && (\lambda \in \Lambda, i \in I),\\
&k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda,\alpha_i)} f_i && (\lambda \in \Lambda, i \in I),\\
&e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} && (i, j \in I),\\
&\sum_{n=0}^{1-a_{ij}} (-1)^n e_i^{(1-a_{ij}-n)} e_j e_i^{(n)} = 0 && (i, j \in I,\, i \neq j),\\
&\sum_{n=0}^{1-a_{ij}} (-1)^n f_i^{(1-a_{ij}-n)} f_j f_i^{(n)} = 0 && (i, j \in I,\, i \neq j),
\end{aligned}
$$

where $q_i = q^{(\alpha_i,\alpha_i)/2}, k_i = k_{\alpha_i}, a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ for $i, j \in I$, and

$$e_i^{(n)} = e_i^n/[n]_{q_i}!, \qquad f_i^{(n)} = f_i^n/[n]_{q_i}!$$

for $i \in I$ and $n \in \mathbb{Z}_{\geqq 0}$. We will use the Hopf algebra structure of $U_{\mathbb{F}}$ given by

$$
\begin{aligned}
&\Delta(k_\lambda) = k_\lambda \otimes k_\lambda && (\lambda \in \Lambda),\\
&\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i && (i \in I),\\
&\varepsilon(k_\lambda) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0 && (\lambda \in \Lambda, i \in I),\\
&S(k_\lambda) = k_\lambda^{-1}, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i && (\lambda \in \Lambda, i \in I).
\end{aligned}
$$

Define subalgebras $U_{\mathbb{F}}^0$, $U_{\mathbb{F}}^+$, $U_{\mathbb{F}}^-$, $U_{\mathbb{F}}^{\geqq 0}$, $U_{\mathbb{F}}^{\leqq 0}$ of $U_{\mathbb{F}}$ by

$$
\begin{aligned}
&U_{\mathbb{F}}^0 = \langle k_\lambda \mid \lambda \in \Lambda \rangle, \qquad U_{\mathbb{F}}^+ = \langle e_i \mid i \in I \rangle, \qquad U_{\mathbb{F}}^- = \langle f_i \mid i \in I \rangle,\\
&U_{\mathbb{F}}^{\geqq 0} = \langle k_\lambda, e_i \mid \lambda \in \Lambda, i \in I \rangle, \qquad U_{\mathbb{F}}^{\leqq 0} = \langle k_\lambda, f_i \mid \lambda \in \Lambda, i \in I \rangle.
\end{aligned}
$$

The multiplication of $U_{\mathbb{F}}$ induces isomorphisms

$$U_{\mathbb{F}} \cong U_{\mathbb{F}}^- \otimes U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^+ \cong U_{\mathbb{F}}^+ \otimes U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^-, \tag{1.1}$$

$$U_{\mathbb{F}}^{\geqq 0} \cong U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^+ \cong U_{\mathbb{F}}^+ \otimes U_{\mathbb{F}}^0, \tag{1.2}$$

$$U_{\mathbb{F}}^{\leqq 0} \cong U_{\mathbb{F}}^0 \otimes U_{\mathbb{F}}^- \cong U_{\mathbb{F}}^- \otimes U_{\mathbb{F}}^0, \tag{1.3}$$

of $\mathbb{F}$-modules. (1.1) is called the triangular decomposition of $U_{\mathbb{F}}$. For $\gamma \in Q$ we set

$$U_{\mathbb{F},\gamma}^{\pm} = \{u \in U_{\mathbb{F}}^{\pm} \mid k_\mu u k_{-\mu} = q^{(\gamma,\mu)} u \quad (\mu \in \Lambda)\}.$$

Then we have

$$U_{\mathbb{F}}^{\pm} = \bigoplus_{\gamma \in Q^+} U_{\mathbb{F},\pm\gamma}^{\pm}.$$

For $i \in I$ we denote by T_i the automorphism of the algebra $U_{\mathbb{F}}$ given by

$$T_i(k_\mu) = k_{s_i\mu} \qquad (\mu \in \Lambda),$$
$$T_i(e_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} e_i^{(-a_{ij}-k)} e_j e_i^{(k)} & (j \in I,\ j \neq i), \\ -f_i k_i & (j = i), \end{cases}$$
$$T_i(f_j) = \begin{cases} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{k} f_i^{(k)} f_j f_i^{(-a_{ij}-k)} & (j \in I,\ j \neq i), \\ -k_i^{-1} e_i & (j = i) \end{cases}$$

(see Lusztig [15]). Let w_0 be the longest element of W. We fix a reduced expression

$$w_0 = s_{i_1} \cdots s_{i_N}$$

of w_0, and set

$$\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \qquad (1 \leqq k \leqq N).$$

Then we have $\Delta^+ = \{\beta_k \mid 1 \leqq k \leqq N\}$. For $1 \leqq k \leqq N$ set

$$e_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(e_{i_k}), \quad f_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f_{i_k}).$$

Then $\{e_{\beta_N}^{m_N} \cdots e_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$ (resp. $\{f_{\beta_N}^{m_N} \cdots f_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$) is an $\mathbb{F}$-basis of $U_{\mathbb{F}}^+$ (resp. $U_{\mathbb{F}}^-$), called the PBW-basis (see Lusztig [14]). We have $e_{\alpha_i} = e_i$ and $f_{\alpha_i} = f_i$ for any $i \in I$.

Denote by

$$\tau : U_{\mathbb{F}}^{\geqq 0} \times U_{\mathbb{F}}^{\leqq 0} \to \mathbb{F} \tag{1.4}$$

the Drinfeld paring. It is characterized as the unique bilinear form satisfying

$$\tau(x, y_1y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \qquad (x \in U_{\mathbb{F}}^{\geqq 0},\ y_1, y_2 \in U_{\mathbb{F}}^{\leqq 0}),$$
$$\tau(x_1x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \qquad (x_1, x_2 \in U_{\mathbb{F}}^{\geqq 0},\ y \in U_{\mathbb{F}}^{\leqq 0}),$$
$$\tau(k_\lambda, k_\mu) = q^{-(\lambda,\mu)} \qquad (\lambda, \mu \in \Lambda),$$
$$\tau(k_\lambda, f_i) = \tau(e_i, k_\lambda) = 0 \qquad (\lambda \in \Lambda,\ i \in I),$$
$$\tau(e_i, f_j) = \delta_{ij}/(q_i^{-1} - q_i) \qquad (i, j \in I)$$

(see [15], [18]). It satisfies the following (see [15], [18]).

Lemma **1.1.** (i) *$\tau(S(x), S(y)) = \tau(x, y)$ for $x \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}$.*

(ii) *For $x \in U_{\mathbb{F}}^{\geqq 0}, y \in U_{\mathbb{F}}^{\leqq 0}$ we have*

$$yx = \sum_{(x)_2, (y)_2} \tau(x_{(0)}, S(y_{(0)}))\tau(x_{(2)}, y_{(2)}) x_{(1)} y_{(1)},$$
$$xy = \sum_{(x)_2, (y)_2} \tau(x_{(0)}, y_{(0)})\tau(x_{(2)}, S(y_{(2)})) y_{(1)} x_{(1)}.$$

(iii) *$\tau(xk_\lambda, yk_\mu) = q^{-(\lambda,\mu)}\tau(x, y)$ for $\lambda, \mu \in \Lambda, x \in U_{\mathbb{F}}^+, y \in U_{\mathbb{F}}^-$.*

(iv) $\tau(U^+_{\mathbb{F},\beta}, U^-_{\mathbb{F},-\gamma}) = \{0\}$ *for* $\beta, \gamma \in Q^+$ *with* $\beta \neq \gamma$.
(v) *For any* $\beta \in Q^+$ *the restriction* $\tau|_{U^+_{\mathbb{F},\beta} \times U^-_{\mathbb{F},-\beta}}$ *is non-degenerate.*

We define an algebra homomorphism

$$\mathrm{ad} : U_{\mathbb{F}} \to \mathrm{End}_{\mathbb{F}}(U_{\mathbb{F}})$$

by

$$\mathrm{ad}(u)(v) = \sum_{(u)} u_{(0)} v(S u_{(1)}) \qquad (u, v \in U_{\mathbb{F}}).$$

1.1.3. We fix an integer $\ell > 1$ satisfying

(a) ℓ is odd,
(b) ℓ is prime to 3 if G is of type G_2,
(c) ℓ is prime to $|\Lambda/Q|$,

and a primitive ℓ-th root $\zeta' \in \mathbb{C}$ of 1. Define a subring $\mathbb{A}$ of $\mathbb{F}$ by

$$\mathbb{A} = \{f(q^{1/|\Lambda/Q|}) \mid f(x) \in \mathbb{Q}(x),\ f \text{ is regular at } x = \zeta'\}.$$

We set $\zeta = (\zeta')^{|\Lambda/Q|}$. We note that ζ is also a primitive ℓ-th root of 1 by the condition (c).

We denote by $U^L_{\mathbb{A}}$, $U_{\mathbb{A}}$ the $\mathbb{A}$-forms of $U_{\mathbb{F}}$ called the Lusztig form and the De Concini-Kac form respectively. Namely, we have

$$U^L_{\mathbb{A}} = \langle e_i^{(m)},\ f_i^{(m)},\ k_\lambda \mid i \in I,\ m \in \mathbb{Z}_{\geqq 0},\ \lambda \in \Lambda\rangle_{\mathbb{A}-\mathrm{alg}} \subset U_{\mathbb{F}},$$
$$U_{\mathbb{A}} = \langle e_i,\ f_i,\ k_\lambda \mid i \in I,\ \lambda \in \Lambda\rangle_{\mathbb{A}-\mathrm{alg}} \subset U_{\mathbb{F}}.$$

We have obviously $U_{\mathbb{A}} \subset U^L_{\mathbb{A}}$. The Hopf algebra structure of $U_{\mathbb{F}}$ induces Hopf algebra structures over $\mathbb{A}$ of $U^L_{\mathbb{A}}$ and $U_{\mathbb{A}}$. We set

$$U^{L,\flat}_{\mathbb{A}} = U^L_{\mathbb{A}} \cap U^\flat_{\mathbb{F}}, \quad U^\flat_{\mathbb{A}} = U_{\mathbb{A}} \cap U^\flat_{\mathbb{F}} \qquad (\flat = +, -, 0, \geqq 0, \leqq 0),$$
$$U^{L,\pm}_{\mathbb{A},\pm\gamma} = U^L_{\mathbb{A}} \cap U^\pm_{\mathbb{F},\pm\gamma}, \quad U^\pm_{\mathbb{A},\pm\gamma} = U_{\mathbb{A}} \cap U^\pm_{\mathbb{F},\pm\gamma} \qquad (\gamma \in Q^+).$$

Then we have triangular decompositions

$$U^L_{\mathbb{A}} \cong U^{L,-}_{\mathbb{A}} \otimes_{\mathbb{A}} U^{L,0}_{\mathbb{A}} \otimes_{\mathbb{A}} U^{L,+}_{\mathbb{A}},$$
$$U_{\mathbb{A}} \cong U^-_{\mathbb{A}} \otimes_{\mathbb{A}} U^0_{\mathbb{A}} \otimes_{\mathbb{A}} U^+_{\mathbb{A}}.$$

Moreover, we have

$$U^{L,\pm}_{\mathbb{A}} = \bigoplus_{\gamma \in Q^+} U^{L,\pm}_{\mathbb{A},\pm\gamma}, \qquad U^\pm_{\mathbb{A}} = \bigoplus_{\gamma \in Q^+} U^\pm_{\mathbb{A},\pm\gamma}.$$

The Drinfeld paring (1.4) induces

$$ {}^L\tau_{\mathbb{A}} : U^{L,\geqq 0}_{\mathbb{A}} \times U^{\leqq 0}_{\mathbb{A}} \to \mathbb{A}, \qquad \tau^L_{\mathbb{A}} : U^{\geqq 0}_{\mathbb{A}} \times U^{L,\leqq 0}_{\mathbb{A}} \to \mathbb{A}. \tag{1.5}$$

Lemma **1.2.** $\mathrm{ad}(U^L_{\mathbb{A}})(U_{\mathbb{A}}) \subset U_{\mathbb{A}}$.

Proof. It is sufficient to show

(1.6) $\mathrm{ad}(k_\lambda)(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \qquad (\lambda \in \Lambda),$

(1.7) $\mathrm{ad}(e_i^{(n)})(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \qquad (i \in I,\, n \in \mathbb{Z}_{\geqq 0}),$

(1.8) $\mathrm{ad}(f_i^{(n)})(U_{\mathbb{A}}) \subset U_{\mathbb{A}} \qquad (i \in I,\, n \in \mathbb{Z}_{\geqq 0}).$

The proof of (1.6) is easy and omitted. By the formulas

$$\mathrm{ad}(x)(uv) = \sum_{(x)} \mathrm{ad}(x_{(0)})(u)\,\mathrm{ad}(x_{(1)})(v) \qquad (x \in U^L_{\mathbb{A}},\, u, v \in U_{\mathbb{A}}),$$

$$\Delta(e_i^{(n)}) = \sum_{r=0}^{n} q_i^{r(n-r)} e_i^{(n-r)} k_i^r \otimes e_i^{(r)} \qquad (i \in I, n \geqq 0),$$

$$\Delta(f_i^{(n)}) = \sum_{r=0}^{n} q_i^{-r(n-r)} f_i^{(r)} \otimes k_i^{-r} f_i^{(n-r)} \qquad (i \in I, n \geqq 0)$$

we have only to show

(1.9) $\mathrm{ad}(e_i^{(n)})(u) \in U_{\mathbb{A}} \qquad (i \in I,\, n \in \mathbb{Z}_{\geqq 0},\, u = k_\lambda, e_j, f_j k_j),$

(1.10) $\mathrm{ad}(f_i^{(n)})(u) \in U_{\mathbb{A}} \qquad (i \in I,\, n \in \mathbb{Z}_{\geqq 0},\, u = k_\lambda, e_j, f_j).$

For $\lambda \in \Lambda$, $i, j \in I$ with $i \neq j$ and $n \in \mathbb{Z}_{>0}$ we have

$$\mathrm{ad}(e_i^{(n)})(k_\lambda) = \frac{(-1)^n q_i^{n(n-1)}}{[n]_{q_i}!} \left(\prod_{j=0}^{n-1} (q_i^{(\lambda, \alpha_i^\vee)} - q_i^{-2j}) \right) e_i^n k_\lambda,$$

$$\mathrm{ad}(e_i^{(n)})(e_i) = q_i^{-n(n+1)/2} (q_i - q_i^{-1})^n e_i^{n+1},$$

$$\mathrm{ad}(e_i^{(n)})(e_j) = \begin{cases} \sum_{r=0}^{n} (-1)^r q_i^{r(n-1+a_{ij})} e_i^{(n-r)} e_j e_i^{(r)} & (n < 1 - a_{ij}), \\ 0 & (n \geqq 1 - a_{ij}), \end{cases}$$

$$\mathrm{ad}(e_i^{(n)})(f_i k_i) = \begin{cases} (k_i^2 - 1)/(q_i - q_i^{-1}) & (n = 1), \\ (-1)^{n-1} q_i^{(n-1)(n+2)/2} (q_i - q_i^{-1})^{n-2} e_i^{n-1} k_i^2 & (n > 1), \end{cases}$$

$$\mathrm{ad}(e_i^{(n)})(f_j k_j) = 0,$$

and hence (1.9) holds (note that $[r]_{q_i}!$ is invertible in $\mathbb{A}$ for $r \leqq -a_{ij}$). The proof of (1.10) is similar and omitted. $\square$

1.1.4. Let us consider the specialization

$$\mathbb{A} \to \mathbb{C} \qquad (q^{1/|\Lambda/Q|} \mapsto \zeta').$$

Note that q is mapped to $\zeta = (\zeta')^{|\Lambda/Q|} \in \mathbb{C}$, which is also a primitive ℓ-th root of 1. We set

$$U^L_\zeta = \mathbb{C} \otimes_{\mathbb{A}} U^L_{\mathbb{A}}, \qquad U_\zeta = \mathbb{C} \otimes_{\mathbb{A}} U_{\mathbb{A}},$$

$$U^{L,\flat}_\zeta = \mathbb{C} \otimes_{\mathbb{A}} U^{L,\flat}_{\mathbb{A}}, \qquad U^{\flat}_\zeta = \mathbb{C} \otimes_{\mathbb{A}} U^{\flat}_{\mathbb{A}} \qquad (\flat = +, -, 0, \geqq 0, \leqq 0),$$

$$U^{L,\pm}_{\zeta,\pm\gamma} = \mathbb{C} \otimes_{\mathbb{A}} U^{L,\pm}_{\mathbb{A},\pm\gamma}, \qquad U^{\pm}_{\zeta,\pm\gamma} = \mathbb{C} \otimes_{\mathbb{A}} U^{\pm}_{\mathbb{A},\pm\gamma} \qquad (\gamma \in Q^+).$$

Then U_ζ^L and U_ζ are Hopf algebras over $\mathbb{C}$, and we have triangular decompositions

$$\begin{aligned} &U_\zeta^L \cong U_\zeta^{L,-} \otimes U_\zeta^{L,0} \otimes U_\zeta^{L,+}, \\ &U_\zeta \cong U_\zeta^- \otimes U_\zeta^0 \otimes U_\zeta^+. \end{aligned}$$

Moreover, we have

$$U_\zeta^{L,\pm} = \bigoplus_{\gamma \in Q^+} U_{\zeta,\pm\gamma}^{L,\pm}, \qquad U_\zeta^{\pm} = \bigoplus_{\gamma \in Q^+} U_{\zeta,\pm\gamma}^{\pm}.$$

By De Concini-Kac [7] we have the following.

Lemma 1.3. (i) $\{e_{\beta_N}^{m_N} \cdots e_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$ *(resp.* $\{f_{\beta_N}^{m_N} \cdots f_{\beta_1}^{m_1} \mid m_1, \dots, m_N \geqq 0\}$*) is a* $\mathbb{C}$*-basis of* U_ζ^+ *(resp.* U_ζ^-*).*

(ii) $\{k_\lambda \mid \lambda \in \Lambda\}$ *is a* $\mathbb{C}$*-basis of* U_ζ^0 *.*

The Drinfeld parings (1.5) induce

$$(1.11) \qquad {}^L\tau_\zeta : U_\zeta^{L,\geqq 0} \times U_\zeta^{\leqq 0} \to \mathbb{C}, \qquad \tau_\zeta^L : U_\zeta^{\geqq 0} \times U_\zeta^{L,\leqq 0} \to \mathbb{C}.$$

Moreover, we have the following (see [20, Lemma 1.5]).

Proposition 1.4. *For any* $\gamma \in Q^+$ *the restrictions of* ${}^L\tau_\zeta$ *and* τ_ζ^L *to*

$$U_{\zeta,\gamma}^{L,+} \times U_{\zeta,-\gamma}^- \to \mathbb{C}, \qquad U_{\zeta,\gamma}^- \times U_{\zeta,-\gamma}^{L,-} \to \mathbb{C}$$

respectively are non-degenerate.

By Lemma 1.2 we have an algebra homomorphism

$$\mathrm{ad} : U_\zeta^L \to \mathrm{End}_{\mathbb{C}}(U_\zeta).$$

In general for a Lie algebra $\mathfrak{s}$ we denote its enveloping algebra by $U(\mathfrak{s})$. We denote by

$$(1.12) \qquad \pi : U_\zeta^L \to U(\mathfrak{g})$$

Lusztig's Frobenius homomorphism ([14]). Namely π is the $\mathbb{C}$-algebra homomorphism given by

$$\pi(e_i^{(m)}) = \begin{cases} \overline{e}_i^{(m/\ell)} & (\ell | m) \\ 0 & (\ell \not| m), \end{cases} \quad \pi(f_i^{(m)}) = \begin{cases} \overline{f}_i^{(m/\ell)} & (\ell | m) \\ 0 & (\ell \not| m), \end{cases} \quad \pi(k_\lambda) = 1$$

for $i \in I$, $m \in \mathbb{Z}_{\geqq 0}$, $\lambda \in \Lambda$. Here, $\overline{e}_i^{(n)} = \overline{e}_i^n/n!$, $\overline{f}_i^{(n)} = \overline{f}_i^n/n!$ for $i \in I$ and $n \in \mathbb{Z}_{\geqq 0}$. Then π is a homomorphism of Hopf algebras.

We recall the description of the center $Z(U_\zeta)$ of the algebra U_ζ due to De Concini-Kac [7] and De Concini-Procesi [8]. Denote by $Z(U_{\mathbb{F}})$ the center of $U_{\mathbb{F}}$, and define a subalgebra $Z_{Har}(U_\zeta)$ of $Z(U_\zeta)$ by

$$Z_{Har}(U_\zeta) = \mathrm{Im}(Z(U_{\mathbb{F}}) \cap U_{\mathbb{A}} \to U_\zeta).$$

We define a shifted action of W on the group algebra $\mathbb{C}[\Lambda] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}e(\lambda)$ of Λ by

$$(1.13) \qquad w \circ e(\lambda) = \zeta^{(w\lambda - \lambda, \rho)} e(w\lambda) \qquad (w \in W,\ \lambda \in \Lambda).$$

Let

$$\iota : Z_{Har}(U_\zeta) \to \mathbb{C}[\Lambda] \tag{1.14}$$

be the composite of

$$Z_{Har}(U_\zeta) \hookrightarrow U_\zeta \cong U_\zeta^- \otimes U_\zeta^0 \otimes U_\zeta^+ \xrightarrow{\varepsilon\otimes 1\otimes\varepsilon} U_\zeta^0 \cong \mathbb{C}[\Lambda],$$

where $U_\zeta^0 \cong \mathbb{C}[\Lambda]$ is given by $k_\lambda \leftrightarrow e(\lambda)$. Then by [7] ι is an injective algebra homomorphism with image

$$\mathbb{C}[2\Lambda]^{W\circ} = \{f \in \mathbb{C}[2\Lambda] \mid w \circ f = f \quad (\forall w \in W)\}.$$

In particular, we have an isomorphism

$$Z_{Har}(U_\zeta) \simeq \mathbb{C}[2\Lambda]^{W\circ} \tag{1.15}$$

of $\mathbb{C}$-algebras. By [7] the elements

$$e_\beta{}^\ell, \quad f_\beta{}^\ell, \quad k_{\ell\lambda} \qquad (\beta \in \Delta^+,\ \lambda \in \Lambda)$$

are central in U_ζ. Let $Z_{Fr}(U_\zeta)$ be the subalgebra of U_ζ generated by them. It is a Hopf subalgebra of U_ζ. Define an algebraic subgroup K of $B^+ \times B^-$ by

$$K = \{(gh, g'h^{-1}) \mid h \in H,\ g \in N^+,\ g' \in N^-\}.$$

By [8] we have an isomorphism

$$Z_{Fr}(U_\zeta) \cong \mathbb{C}[K] \tag{1.16}$$

of Hopf algebras. We refer the reader to [20] for the explicit description of the isomorphism (1.16). By [8] $Z(U_\zeta)$ is generated by $Z_{Fr}(U_\zeta)$ and $Z_{Har}(U_\zeta)$. Moreover, we have an isomorphism

$$Z(U_\zeta) \cong Z_{Har}(U_\zeta) \otimes_{Z_{Har}(U_\zeta)\cap Z_{Fr}(U_\zeta)} Z_{Fr}(U_\zeta) \qquad (z_1z_2 \leftrightarrow z_1 \otimes z_2)$$

of algebras.

1.2. **Sheaves on quantized flag manifolds.**

1.2.1. We denote by $C_{\mathbb{F}}$ the subspace of $U_{\mathbb{F}}^* = \mathrm{Hom}_{\mathbb{F}}(U_{\mathbb{F}}, \mathbb{F})$ spanned by the matrix coefficients of finite-dimensional $U_{\mathbb{F}}$-modules of type 1 in the sense of Lusztig, and denote by

$$\langle\ ,\ \rangle : C_{\mathbb{F}} \times U_{\mathbb{F}} \to \mathbb{F} \tag{1.17}$$

the canonical paring. Then $C_{\mathbb{F}}$ is endowed with a Hopf algebra structure dual to $U_{\mathbb{F}}$ via (1.17). We have a $U_{\mathbb{F}}$-bimodule structure of $C_{\mathbb{F}}$ given by

$$\langle u_1 \cdot \varphi \cdot u_2, u\rangle = \langle \varphi, u_2uu_1\rangle \qquad (\varphi \in C_{\mathbb{F}}, u, u_1, u_2 \in U_{\mathbb{F}}).$$

Define a Λ-graded ring $A_{\mathbb{F}} = \bigoplus_{\lambda\in\Lambda^+} A_{\mathbb{F}}(\lambda)$ by

$$A_{\mathbb{F}} = \{\varphi \in C_{\mathbb{F}} \mid \varphi \cdot f_i = 0 \quad (i \in I)\},$$
$$A_{\mathbb{F}}(\lambda) = \{\varphi \in A_{\mathbb{F}} \mid \varphi \cdot k_\mu = q^{(\mu,\lambda)}\varphi \quad (\mu \in \Lambda)\}.$$

Note that $A_{\mathbb{F}}$ is a left $U_{\mathbb{F}}$-submodule of $C_{\mathbb{F}}$. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$ we set

$$A_{\mathbb{F}}(\lambda)_\xi = \{\varphi \in A_{\mathbb{F}}(\lambda) \mid k_\mu \cdot \varphi = q^{(\xi,\mu)}\varphi\}.$$

Then we have

$$A_{\mathbb{F}}(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_{\mathbb{F}}(\lambda)_\xi.$$

We define $\mathbb{A}$-forms $C_{\mathbb{A}}$, $A_{\mathbb{A}}$, $A_{\mathbb{A}}(\lambda)$ ($\lambda \in \Lambda^+$) of $C_{\mathbb{F}}$, $A_{\mathbb{F}}$, $A_{\mathbb{F}}(\lambda)$ respectively by

$$C_{\mathbb{A}} = \{\varphi \in C_{\mathbb{F}} \mid \langle \varphi, U_{\mathbb{A}}^L \rangle \subset \mathbb{A}\}, \quad A_{\mathbb{A}} = A_{\mathbb{F}} \cap C_{\mathbb{A}}, \quad A_{\mathbb{A}}(\lambda) = A_{\mathbb{F}}(\lambda) \cap C_{\mathbb{A}}.$$

Then $C_{\mathbb{A}}$ is a Hopf algebra over $\mathbb{A}$, and $A_{\mathbb{A}}$ is its $\mathbb{A}$-subalgebra. Moreover, $C_{\mathbb{A}}$ is a $U_{\mathbb{A}}^L$-bimodule and $A_{\mathbb{A}}$ is its left $U_{\mathbb{A}}^L$-submodule. We also set $A_{\mathbb{A}}(\lambda)_\xi = A_{\mathbb{F}}(\lambda)_\xi \cap A_{\mathbb{A}}$ for $\lambda \in \Lambda^+$, $\xi \in \Lambda$.

We set

$$C_\zeta = \mathbb{C} \otimes_{\mathbb{A}} C_{\mathbb{A}}, \quad A_\zeta = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}, \quad A_\zeta(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\lambda) \qquad (\lambda \in \Lambda^+).$$

Then C_ζ is a Hopf algebra over $\mathbb{C}$. Moreover, the $U_{\mathbb{F}}$-bimodule structure of $C_{\mathbb{F}}$ induces a U_ζ^L-bimodule structure of C_ζ. For $\lambda \in \Lambda^+$ and $\xi \in \Lambda$ we set $A_\zeta(\lambda)_\xi = \mathbb{C} \otimes_{\mathbb{A}} A_{\mathbb{A}}(\lambda)_\xi$ Then we have

$$A_\zeta(\lambda) = \bigoplus_{\xi \in \lambda - Q^+} A_\zeta(\lambda)_\xi.$$

We have a natural paring

$$\langle\ ,\ \rangle : C_\zeta \times U_\zeta^L \to \mathbb{C}. \tag{1.18}$$

induced by (1.17).

1.2.2. For a ring (resp. Λ-graded ring) $\mathcal{R}$ we denote by $\mathrm{Mod}(\mathcal{R})$ (resp. $\mathrm{Mod}_\Lambda(\mathcal{R})$) the category of $\mathcal{R}$-modules (resp. Λ-graded left $\mathcal{R}$-modules). Assume that we are given a homomorphism $\jmath : A \to B$ of Λ-graded rings satisfying

$$\jmath(A(\lambda))B(\mu) = B(\mu)\jmath(A(\lambda)) \qquad (\lambda, \mu \in \Lambda). \tag{1.19}$$

For $M \in \mathrm{Mod}_\Lambda(B)$ let $\mathrm{Tor}(M)$ be the subset of M consisting of $m \in M$ such that there exists $\lambda \in \Lambda^+$ satisfying $\jmath(A(\lambda + \mu))m = \{0\}$ for any $\mu \in \Lambda^+$. Then $\mathrm{Tor}(M)$ is a subobject of M in $\mathrm{Mod}_\Lambda(B)$ by (1.19). We denote by $\mathrm{Tor}_{\Lambda^+}(A, B)$ the full subcategory of $\mathrm{Mod}_\Lambda(B)$ consisting of $M \in \mathrm{Mod}_\Lambda(B)$ such that $\mathrm{Tor}(M) = M$. Note that $\mathrm{Tor}_{\Lambda^+}(A, B)$ is closed under taking subquotients and extensions in $\mathrm{Mod}_\Lambda(B)$. Let $\Sigma(A, B)$ denote the collection of morphisms f of $\mathrm{Mod}_\Lambda(B)$ such that its kernel $\mathrm{Ker}(f)$ and its cokernel $\mathrm{Coker}(f)$ belong to $\mathrm{Tor}_{\Lambda^+}(A, B)$. Then we define an abelian category $\mathcal{C}(A, B) = \mathrm{Mod}_\Lambda(B)/\mathrm{Tor}_{\Lambda^+}(A, B)$ as the localization

$$\mathcal{C}(A, B) = \Sigma(A, B)^{-1}\,\mathrm{Mod}_\Lambda(B)$$

of $\mathrm{Mod}_\Lambda(B)$ with respect to the multiplicative system $\Sigma(A,B)$ (see, for example, Popescu [16] for the notion of localization of categories). We denote by

$$\omega(A,B)^* : \mathrm{Mod}_\Lambda(B) \to \mathcal{C}(A,B) \tag{1.20}$$

the canonical exact functor. It admits a right adjoint

$$\omega(A,B)_* : \mathcal{C}(A,B) \to \mathrm{Mod}_\Lambda(B), \tag{1.21}$$

which is left exact. It is known that $\omega(A,B)^* \circ \omega(A,B)_* \cong \mathrm{Id}$. By taking the degree zero part of (1.21) we obtain a left exact functor

$$\Gamma_{(A,B)} : \mathcal{C}(A,B) \to \mathrm{Mod}(B(0)). \tag{1.22}$$

The abelian category $\mathcal{C}(A,B)$ has enough injectives, and we have the right derived functors

$$R^i\Gamma_{(A,B)} : \mathcal{C}(A,B) \to \mathrm{Mod}(B(0)) \qquad (i \in \mathbb{Z}) \tag{1.23}$$

of (1.22).

We apply the above arguments to the case $A = B = A_\zeta$. Then $\mathrm{Tor}(M)$ for $M \in \mathrm{Mod}_\Lambda(A_\zeta)$ consists of $m \in M$ such that there exists $\lambda \in \Lambda^+$ satisfying $A_\zeta(\lambda)m = \{0\}$ (see [20, Lemma 3.4]). We set

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) = \mathcal{C}(A_\zeta, A_\zeta). \tag{1.24}$$

In this case the natural functors (1.20), (1.21), (1.22) are simply denoted as

$$\omega^* : \mathrm{Mod}_\Lambda(A_\zeta) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}), \tag{1.25}$$

$$\omega_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}_\Lambda(A_\zeta), \tag{1.26}$$

$$\Gamma : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C}). \tag{1.27}$$

Remark **1.5.** In the terminology of non-commutative algebraic geometry $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ is the category of "quasi-coherent sheaves" on the quantized flag manifold $\mathcal{B}_\zeta$, which is a "non-commutative projective scheme". The notations $\mathcal{B}_\zeta$, $\mathcal{O}_{\mathcal{B}_\zeta}$ have only symbolical meaning.

1.2.3. Using Lusztig's Frobenius homomorphism (1.12) we will relate the quantized flag manifold $\mathcal{B}_\zeta$ with the ordinary flag manifold $\mathcal{B} = B^-\backslash G$. Taking the dual Hopf algebras in (1.12) we obtain an injective homomorphism $\mathbb{C}[G] \to C_\zeta$ of Hopf algebras. Moreover, its image is contained in the center of C_ζ (see Lusztig [14]). We will regard $\mathbb{C}[G]$ as a central Hopf subalgebra of C_ζ in the following. Setting

$$\begin{aligned} A_1 =& \{\varphi \in \mathbb{C}[G] \mid \varphi(ng) = \varphi(g) \ (n \in N^-,\, g \in G)\}, \\ A_1(\lambda) =& \{\varphi \in A_1 \mid \varphi(tg) = \theta_\lambda(t)\varphi(g) \ (t \in H,\, g \in G)\} \qquad (\lambda \in \Lambda^+) \end{aligned}$$

we have a Λ-graded algebra

$$A_1 = \bigoplus_{\lambda \in \Lambda^+} A_1(\lambda).$$

We have a left G-module structure of A_1 given by

$$(x\varphi)(g) = \varphi(gx) \qquad (\varphi \in A_1, x, g \in G).$$

In particular, A_1 is a $U(\mathfrak{g})$-module. Moreover, for each $\lambda \in \Lambda^+$, $A_1(\lambda)$ is a $U(\mathfrak{g})$-submodule of A_1 which is an irreducible highest weight module with highest weight λ. Regarding $\mathbb{C}[G]$ as a subalgebra of C_ζ we have

$$A_1 = A_\zeta \cap \mathbb{C}[G], \qquad A_1(\lambda) = A_\zeta(\ell\lambda) \cap \mathbb{C}[G].$$

Since the Λ-graded algebra A_1 is the homogeneous coordinate algebra of the projective variety $\mathcal{B} = B^- \backslash G$, we have an identification

$$\mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) = \mathcal{C}(A_1, A_1) \tag{1.28}$$

of abelian categories, where $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$ denotes the category of quasi-coherent $\mathcal{O}_{\mathcal{B}}$-modules on the ordinary flag manifold $\mathcal{B}$. We set

$$\omega_{\mathcal{B}*} = \omega(A_1, A_1)_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}_\Lambda(A_1). \tag{1.29}$$

For $\lambda \in \Lambda$ we denote by $\mathcal{O}_{\mathcal{B}}(\lambda) \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$ the invertible G-equivariant $\mathcal{O}_{\mathcal{B}}$-module corresponding to λ. Then under the identification (1.28) we have

$$\omega_{\mathcal{B}*}M = \bigoplus_{\lambda \in \Lambda} \Gamma(\mathcal{B}, M \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_{\mathcal{B}}(\lambda)) \qquad (M \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})),$$

where $\Gamma(\mathcal{B}, \ ) : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathbb{C}$ is the global section functor for the algebraic variety $\mathcal{B}$. In particular, the functor $\Gamma_{(A_1,A_1)} : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}(\mathbb{C})$ is identified with $\Gamma(\mathcal{B}, \ )$.

For a Λ-graded $\mathbb{C}$-algebra B we define a new Λ-graded $\mathbb{C}$-algebra $B^{(\ell)}$ by

$$B^{(\ell)}(\lambda) = B(\ell\lambda) \qquad (\lambda \in \Lambda).$$

Let

$$(\ )^{(\ell)} : \mathrm{Mod}_\Lambda(B) \to \mathrm{Mod}_\Lambda(B^{(\ell)}) \tag{1.30}$$

be the exact functor given by

$$M^{(\ell)}(\lambda) = M(\ell\lambda) \qquad (\lambda \in \Lambda)$$

for $M \in \mathrm{Mod}_\Lambda(B)$.

We have the following results ([20]).

Lemma **1.6.** *Let B be a Λ-graded $\mathbb{C}$-algebra. Assume that we are given a homomorphism $\jmath : A_\zeta \to B$ of Λ-graded $\mathbb{C}$-algebras. We denote by $\jmath' : A_1 \to B^{(\ell)}$ the induced homomorphism of Λ-graded $\mathbb{C}$-algebras. Assume*

$$\jmath(A_\zeta(\lambda))B(\mu) = B(\mu)\jmath(A_\zeta(\lambda)) \qquad (\lambda, \mu \in \Lambda),$$
$$\jmath'(A_1(\lambda))B^{(\ell)}(\mu) = B^{(\ell)}(\mu)\jmath'(A_1(\lambda)) \qquad (\lambda, \mu \in \Lambda).$$

Then the exact functor

$$(\ )^{(\ell)} : \mathrm{Mod}_\Lambda(B) \to \mathrm{Mod}_\Lambda(B^{(\ell)})$$

induces an equivalence

$$Fr_* : \mathcal{C}(A_\zeta, B) \to \mathcal{C}(A_1, B^{(\ell)}) \tag{1.31}$$

of abelian categories. Moreover, we have

$$\omega(A_1, B^{(\ell)})_* \circ Fr_* = (\ )^{(\ell)} \circ \omega(A_\zeta, B)_*. \tag{1.32}$$

Lemma **1.7.** *Let F be a Λ-graded $\mathbb{C}$-algebra, and let $A_1 \to F$ be a homomorphism of Λ-graded $\mathbb{C}$-algebras. Assume that $\mathrm{Im}(A_1 \to F)$ is central in F. Regard F as an object of $\mathrm{Mod}_\Lambda(A_1)$ and consider $\omega_{\mathcal{B}}^* F \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}})$. Then the multiplication of F induces an $\mathcal{O}_{\mathcal{B}}$-algebra structure of $\omega_{\mathcal{B}}^* F$, and we have an identification*

$$\mathcal{C}(A_1, F) = \mathrm{Mod}(\omega_{\mathcal{B}}^* F), \tag{1.33}$$

of abelian categories, where $\mathrm{Mod}(\omega_{\mathcal{B}}^* F)$ denotes the category of quasi-coherent $\omega_{\mathcal{B}}^* F$-modules. Moreover, under the identification (1.33) *we have*

$$\Gamma_{(A_1,F)}(M) = \Gamma(\mathcal{B}, M) \in \mathrm{Mod}(F(0)) \qquad (M \in \mathrm{Mod}(\omega_{\mathcal{B}}^* F)).$$

We define an $\mathcal{O}_{\mathcal{B}}$-algebra $Fr_* \mathcal{O}_{\mathcal{B}_\zeta}$ by

$$Fr_* \mathcal{O}_{\mathcal{B}_\zeta} = \omega_{\mathcal{B}}^* (A_\zeta^{(\ell)}).$$

We denote by $\mathrm{Mod}(Fr_* \mathcal{O}_{\mathcal{B}_\zeta})$ the category of quasi-coherent $Fr_* \mathcal{O}_{\mathcal{B}_\zeta}$-modules. By Lemma 1.6 and Lemma 1.7 we have the following.

Lemma **1.8.** *We have an equivalence*

$$Fr_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(Fr_* \mathcal{O}_{\mathcal{B}_\zeta})$$

of abelian categories. Moreover, for $M \in \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ we have

$$R^i\Gamma(M) \simeq R^i\Gamma(\mathcal{B}, Fr_*(M)),$$

where $\Gamma(\mathcal{B},\ ) : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) \to \mathrm{Mod}(\mathbb{C})$ in the right side is the global section functor for $\mathcal{B}$.

2. The category of D-modules

2.1. **Ring of differential operators.**

2.1.1. We define a subalgebra $D_{\mathbb{F}}$ of $\mathrm{End}_{\mathbb{F}}(A_{\mathbb{F}})$ by

$$D_{\mathbb{F}} = \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{F}}, u \in U_{\mathbb{F}}, \lambda \in \Lambda \rangle,$$

where

$$\ell_\varphi(\psi) = \varphi\psi, \quad r_\varphi(\psi) = \psi\varphi, \quad \partial_u(\psi) = u \cdot \psi, \quad \sigma_\lambda(\psi) = q^{(\lambda,\mu)}\psi$$

for $\psi \in A_{\mathbb{F}}(\mu)$. In fact we have

$$D_{\mathbb{F}} = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{F}}, u \in U_{\mathbb{F}}, \lambda \in \Lambda \rangle$$

by [20, Lemma 4.1].

We have a natural grading

$$D_{\mathbb{F}} = \bigoplus_{\lambda \in \Lambda^+} D_{\mathbb{F}}(\lambda),$$
$$D_{\mathbb{F}}(\lambda) = \{\Phi \in D_{\mathbb{F}} \mid \Phi(A_{\mathbb{F}}(\mu)) \subset A_{\mathbb{F}}(\lambda+\mu) \quad (\mu \in \Lambda)\} \qquad (\lambda \in \Lambda)$$

of $D_{\mathbb{F}}$. It is easily checked that

$$\begin{aligned} \partial_u \ell_\varphi &= \sum_{(u)} \ell_{u_{(0)}\cdot\varphi} \partial_{u_{(1)}} && (u \in U_{\mathbb{F}}, \varphi \in A_{\mathbb{F}}), \\ \partial_u \sigma_\lambda &= \sigma_\lambda \partial_u && (u \in U_{\mathbb{F}}, \lambda \in \Lambda), \\ \sigma_\lambda \ell_\varphi &= q^{(\lambda,\mu)} \ell_\varphi \sigma_\lambda && (\lambda \in \Lambda, \varphi \in A_{\mathbb{F}}(\mu)). \end{aligned}$$

Set

$$E_{\mathbb{F}} = A_{\mathbb{F}} \otimes U_{\mathbb{F}} \otimes \mathbb{F}[\Lambda].$$

We have a natural $\mathbb{F}$-algebra structure of $E_{\mathbb{F}}$ such that $A_{\mathbb{F}} \otimes 1 \otimes 1$, $1 \otimes U_{\mathbb{F}} \otimes 1$, $1 \otimes 1 \otimes \mathbb{F}[\Lambda]$ are subalgebras of $E_{\mathbb{F}}$ naturally isomorphic to $A_{\mathbb{F}}$, $U_{\mathbb{F}}$, $\mathbb{F}[\Lambda]$ respectively and that we have the relations

$$\begin{aligned} u\varphi &= \sum_{(u)} (u_{(0)} \cdot \varphi) u_{(1)} && (u \in U_{\mathbb{F}}, \varphi \in A_{\mathbb{F}}), \\ ue(\lambda) &= e(\lambda)u && (u \in U_{\mathbb{F}}, \lambda \in \Lambda), \\ e(\lambda)\varphi &= q^{(\lambda,\mu)} \varphi e(\lambda) && (\lambda \in \Lambda, \varphi \in A_{\mathbb{F}}(\mu)) \end{aligned}$$

in $E_{\mathbb{F}}$. Here, we identify $A_{\mathbb{F}} \otimes 1 \otimes 1$, $1 \otimes U_{\mathbb{F}} \otimes 1$, $1 \otimes 1 \otimes \mathbb{F}[\Lambda]$ with $A_{\mathbb{F}}$, $U_{\mathbb{F}}$, $\mathbb{F}[\Lambda]$ respectively. Then we have a surjective algebra homomorphism

$$E_{\mathbb{F}} \to D_{\mathbb{F}} \tag{2.1}$$

sending $\varphi \in A_{\mathbb{F}}$, $u \in U_{\mathbb{F}}$, $e(\lambda) \in \mathbb{F}[\Lambda]$ $(\lambda \in \Lambda)$ to ℓ_φ, ∂_u, σ_λ respectively. Moreover, $E_{\mathbb{F}}$ has an obvious Λ-grading so that (2.1) preserves the Λ-grading.

2.1.2. Set

$$\begin{aligned} D_{\mathbb{A}} &= \langle \ell_\varphi, r_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{A}}, u \in U_{\mathbb{A}}, \lambda \in \Lambda \rangle_{\mathbb{A}-\mathrm{alg}} \subset D_{\mathbb{F}}, \\ E_{\mathbb{A}} &= A_{\mathbb{A}} \otimes U_{\mathbb{A}} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{A}}. \end{aligned}$$

They are Λ-graded $\mathbb{A}$-subalgebras of $D_{\mathbb{F}}$ and $E_{\mathbb{F}}$ respectively. Again we have

$$D_{\mathbb{A}} = \langle \ell_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in A_{\mathbb{A}}, u \in U_{\mathbb{A}}, \lambda \in \Lambda \rangle_{\mathbb{A}-\mathrm{alg}}.$$

by [20]. In particular, we have a surjective homomorphism

$$E_{\mathbb{A}} \to D_{\mathbb{A}}$$

of Λ-graded algebras. Note that there is a canonical embedding

$$D_{\mathbb{A}} \to \mathrm{End}_{\mathbb{A}}(A_{\mathbb{A}}).$$

2.1.3. We set

$$D_\zeta = \mathbb{C} \otimes_{\mathbb{A}} D_{\mathbb{A}}, \qquad E_\zeta = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A}} = A_\zeta \otimes U_\zeta \otimes \mathbb{C}[\Lambda].$$

D_ζ is a Λ-graded $\mathbb{C}$-algebra generated by elements of the form

$$\ell_\varphi, \ \partial_u, \ \sigma_\lambda \qquad (\varphi \in A_\zeta, \ u \in U_\zeta, \ \lambda \in \Lambda).$$

We have a surjective homomorphism

$$E_\zeta \to D_\zeta$$

of Λ-graded $\mathbb{C}$-algebras.

Lemma **2.1.** *Let $z \in Z_{Har}(U_\zeta)$, and write $\iota(z) = \sum_{\lambda\in\Lambda} c_\lambda k_{2\lambda} \quad (c_\lambda \in \mathbb{C})$. Then we have*

$$\partial_z = \sum_{\lambda\in\Lambda} c_\lambda \sigma_{2\lambda}.$$

Proof. This follows from the corresponding statement over $\mathbb{F}$, which is given in [19] $\square$

Remark **2.2.** The natural algebra homomorphism $D_\zeta \to \mathrm{End}_{\mathbb{C}}(A_\zeta)$ is not injective.

2.1.4. Define an $\mathcal{O}_{\mathcal{B}}$-algebra $Fr_*\mathcal{D}_{\mathcal{B}_\zeta}$ by

$$Fr_*\mathcal{D}_{\mathcal{B}_\zeta} = \omega_{\mathcal{B}}^* D_\zeta^{(\ell)}.$$

We define $ZD_\zeta^{(\ell)}$ to be the central subalgebra of $D_\zeta^{(\ell)}$ generated by the elements of the form

$$\ell_\varphi, \ \partial_u, \ \sigma_\lambda \qquad (\varphi \in A_1, \ u \in Z_{Fr}(U_\zeta), \ \lambda \in \Lambda),$$

and set

$$\mathcal{Z}_\zeta = \omega_{\mathcal{B}}^* ZD_\zeta^{(\ell)}.$$

It is a central subalgebra of $Fr_*\mathcal{D}_{\mathcal{B}_\zeta}$. Define a subvariety $\mathcal{V}$ of $\mathcal{B}\times K\times H$ by

$$\mathcal{V} = \{(B^- g, k, t) \in \mathcal{B} \times K \times H \mid g\kappa(k)g^{-1} \in t^{2\ell}N^-\},$$

where $\kappa : K \to G$ is given by $\kappa(k_1, k_2) = k_1k_2^{-1}$. We denote by

$$p_{\mathcal{V}} : \mathcal{V} \to \mathcal{B}$$

the projection. Now we can state the main results of [20].

Theorem **2.3** ([20])**.** *The $\mathcal{O}_{\mathcal{B}}$-algebra $\mathcal{Z}_\zeta$ is naturally isomorphic to $p_{\mathcal{V}*}\mathcal{O}_{\mathcal{V}}$.*

Define an $\mathcal{O}_{\mathcal{V}}$-algebra $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ by

$$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = p_{\mathcal{V}}^{-1} Fr_*\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{p_{\mathcal{V}}^{-1}p_{\mathcal{V}*}\mathcal{O}_{\mathcal{V}}} \mathcal{O}_{\mathcal{V}}.$$

Theorem **2.4** ([20])**.** *$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ is an Azumaya algebra of rank $\ell^{2|\Delta^+|}$.*

Define

$$\delta : \mathcal{V} \to K \times_{H/W} H$$

by $\delta(B^- g, k, t) = (k, t)$, where $K \to H/W$ is given by $k \mapsto [h]$ where h is an element of H conjugate to the semisimple part of $\kappa(k)$, and $H \to H/W$ is given by $t \mapsto [t^{2\ell}]$.

THEOREM **2.5** ([20]). *For any $(k,t) \in K \times_{H/W} H$, the restriction of $\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}$ to $\delta^{-1}(k,t)$ is a split Azumaya algebra.*

2.2. **Category of D-modules.** We define an abelian category $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ by

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) = \mathcal{C}(A_\zeta, D_\zeta).$$

By Lemma 1.6 and Lemma 1.7 we have an equivalence

$$Fr_* : \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(Fr_* \mathcal{D}_{\mathcal{B}_\zeta}) \tag{2.2}$$

of abelian categories, where $\mathrm{Mod}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta})$ denotes the category of quasi-coherent $Fr_*\mathcal{D}_{\mathcal{B}_\zeta}$-modules. Moreover, for $M \in \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ we have

$$R^i\Gamma_{(A_\zeta, D_\zeta)}(M) = R^i\Gamma(\mathcal{B}, Fr_*(M)) \in \mathrm{Mod}(D_\zeta(0)), \tag{2.3}$$

where $\Gamma(\mathcal{B}, \ )$ in the right side is the global section functor for the ordinary flag variety $\mathcal{B}$.

For $t \in H$ we define an abelian category $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta, t})$ by

$$\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) = \mathrm{Mod}_{\Lambda,t}(D_\zeta)/(\mathrm{Mod}_{\Lambda,t}(D_\zeta) \cap \mathrm{Tor}_{\Lambda^+}(A_\zeta, D_\zeta)),$$

where $\mathrm{Mod}_{\Lambda,t}(D_\zeta)$ is the full subcategory of $\mathrm{Mod}_\Lambda(D_\zeta)$ consisting of $M \in \mathrm{Mod}_\Lambda(D_\zeta)$ so that $\sigma_\lambda|_{M(\mu)} = \theta_\lambda(t)\zeta^{(\lambda,\mu)}\,\mathrm{id}$ for any $\lambda, \mu \in \Lambda$. Then we can regard $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ as a full subcategory of $\mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta})$ (see [19, Lemma 4.6]). Set

$$Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t} = Fr_*\mathcal{D}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t,$$

where $\mathbb{C}_t$ denotes the one-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda) \mapsto \theta_\lambda(t)$ for $\lambda \in \Lambda$. The equivalence (2.2) induces the equivalence

$$Fr_* : \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t}) \to \mathrm{Mod}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}), \tag{2.4}$$

where $\mathrm{Mod}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ denotes the category of quasi-coherent $Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules. In particular, for $M \in \mathrm{Mod}(\mathcal{D}_{\mathcal{B}_\zeta,t})$ we have

$$R^i\Gamma_{(A_\zeta,D_\zeta)}(M) = R^i\Gamma(\mathcal{B}, Fr_*M) \in \mathrm{Mod}(D_{\zeta,t}(0)),$$

where $D_{\zeta,t}(0) = D_\zeta(0)/\sum_{\lambda\in\Lambda} D_\zeta(0)(\sigma_\lambda - \theta_\lambda(t))$.

2.3. **Conjecture.** By Lemma 2.1 the natural algebra homomorphism

$$U_\zeta \otimes_{\mathbb{C}} \mathbb{C}[\Lambda] \to D_\zeta(0)$$

factors through

$$U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}[\Lambda] \to D_\zeta(0),$$

where $Z_{Har}(U_\zeta)$ is identified with $\mathbb{C}[2\Lambda]^{W\circ}$ by (1.15). Hence we have a natural algebra homomorphism

$$U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}[\Lambda] \to \Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta}). \tag{2.5}$$

For $t \in H$ we denote by $\mathbb{C}_t$ the one-dimensional $\mathbb{C}[\Lambda]$-module given by $e(\lambda)v = \theta_\lambda(t)v$ $(v \in \mathbb{C}_t)$. Then (2.5) induces an algebra homomorphism

$$U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t \to \Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}), \tag{2.6}$$

where $\mathbb{C}_t$ is regarded as a $Z_{Har}(U_\zeta)$-module by $Z_{Har}(U_\zeta) \cong \mathbb{C}[2\Lambda]^{W\circ} \subset \mathbb{C}[\Lambda]$. Denote by h_G the Coxeter number for G.

Conjecture **2.6.** *Assume $\ell > h_G$. The algebra homomorphism* (2.5) *is an isomorphism, and we have*

$$R^i\Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta}) = 0$$

for $i \neq 0$.

Proposition **2.7.** *Let $\ell > h_G$, and assume that Conjecture 2.6 is valid. Then for $t \in H$ we have*

$$\Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}) \cong U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t,$$

and

$$R^i\Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}) = 0 \qquad (i \neq 0).$$

Proof. Define $f : \mathcal{V} \to H$ to be the composite of the embedding $\mathcal{V} \to \mathcal{B}\times K\times H$ and the projection $\mathcal{B}\times K\times H \to H$ onto the third factor. Since $p_\mathcal{V}$ is an affine morphism, we have $Rp_{\mathcal{V}*}\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = p_{\mathcal{V}*}\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} = Fr_*\mathcal{D}_{\mathcal{B}_\zeta}$. Hence we have

$$U_\zeta \otimes^L_{Z_{Har}(U_\zeta)} \mathbb{C}[\Lambda] = U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}[\Lambda] \cong R\Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta}) = R\Gamma(\mathcal{V}, \tilde{\mathcal{D}}_{\mathcal{B}_\zeta}).$$

Here we used the fact that $\mathbb{C}[\Lambda]$ is a free $Z_{Har}(U_\zeta)$-module (see Steinberg [17]). Denote by $\mathcal{O}_t$ the $\mathcal{O}_H$-module corresponding to the $\mathbb{C}[\Lambda]$-module $\mathbb{C}_t$. Similarly we have

$$Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t} = p_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t) = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathbb{C}[\Lambda]} \mathbb{C}_t).$$

Since f is flat, we have $Lf^*\mathcal{O}_t = f^*\mathcal{O}_t$. Hence by Theorem 2.4 we have

$$\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t = \tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} f^*\mathcal{O}_t = \tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes_{\mathcal{O}_\mathcal{V}} f^*\mathcal{O}_t.$$

It follows that

$$Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t} = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t) = Rp_{\mathcal{V}*}(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathcal{O}_H} \mathcal{O}_t.$$

Hence we have

$$\begin{aligned}&R\Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}) = R\Gamma(H, Rf_*(\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_\mathcal{V}} Lf^*\mathcal{O}_t))\\ =&R\Gamma(H, Rf_*\tilde{\mathcal{D}}_{\mathcal{B}_\zeta} \otimes^L_{\mathcal{O}_H} \mathcal{O}_t) = R\Gamma(H, Rf_*\tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t\\ =&R\Gamma(\mathcal{V}, \tilde{\mathcal{D}}_{\mathcal{B}_\zeta}) \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t = U_\zeta \otimes^L_{Z_{Har}(U_\zeta)} \mathbb{C}[\Lambda] \otimes^L_{\mathbb{C}[\Lambda]} \mathbb{C}_t\\ =&U_\zeta \otimes^L_{Z_{Har}(U_\zeta)} \mathbb{C}_t.\end{aligned}$$

□

2.4. **Derived Beilinson-Bernstein equivalence.** We show that Conjecture 2.6 implies a variant of the Beilinson-Bernstein equivalence for derived categories.

Recall that we have an identification

$$Z_{Har}(U_\zeta) \cong \mathbb{C}[2\Lambda]^{W\circ} \subset \mathbb{C}[2\Lambda] \subset \mathbb{C}[\Lambda].$$

Recall also that we identify $\mathbb{C}[\Lambda]$ with the coordinate algebra $\mathbb{C}[H]$ of H. Set $H^{(2)} = H/\operatorname{Ker}(H \ni t \mapsto t^2 \in H)$, and let $\pi : H \to H^{(2)}$ be the canonical homomorphism. Then we have a natural identification $\mathbb{C}[H^{(2)}] = \mathbb{C}[2\Lambda]$ so that $\pi^* : \mathbb{C}[H^{(2)}] \to \mathbb{C}[H]$ is identified with the inclusion $\mathbb{C}[2\Lambda] \subset \mathbb{C}[\Lambda]$. Denote the isomorphism $H \cong H^{(2)}$ corresponding to $\mathbb{C}[\Lambda] \ni e(\lambda) \leftrightarrow e(2\lambda) \in \mathbb{C}[2\Lambda]$ by $t \leftrightarrow t^{1/2}$. Then we have $\pi(t) = (t^2)^{1/2}$. The shifted action (1.13) of W on $\mathbb{C}[2\Lambda]$ induces an action of W on $H^{(2)}$ given by

$$w \circ t^{1/2} = (w(tt_{2\rho})t_{2\rho}^{-1})^{1/2} \qquad (w \in W, t \in H),$$

where $t_{2\rho} \in H$ is given by $\theta_\mu(t_{2\rho}) = \zeta^{2(\mu,\rho)}$ for any $\mu \in \Lambda$ (note that $2(\mu,\rho) \in \mathbb{Z}$), and $Z_{Har}(U_\zeta)$ is regarded as the coordinate algebra of the quotient variety $(W\circ)\backslash H^{(2)}$. For $t \in H$ we denote by $\chi_t : \mathbb{C}[\Lambda] \to \mathbb{C}$ the corresponding algebra homomorphism. By the above argument we have

$$\chi_{t_1}|_{Z_{Har}(U_\zeta)} = \chi_{t_2}|_{Z_{Har}(U_\zeta)} \quad \Longleftrightarrow \quad (t_1^2)^{1/2} \in W \circ t_2^{1/2}.$$

We say that $t \in H$ is regular if

$$\{w \in W \mid w \circ (t^2)^{1/2} = (t^2)^{1/2}\} = \{1\}.$$

We denote by $\operatorname{Mod}_{coh}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})$ (resp. $\operatorname{Mod}_f(U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t)$) the category of coherent $Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t}$-modules (resp. finitely generated $U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t$-modules). We also denote by $\operatorname{Mod}_{coh,t}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta})$ (resp. $\operatorname{Mod}_{f,t}(U_\zeta)$) the category of coherent $Fr_*\mathcal{D}_{\mathcal{B}_\zeta}$-modules (resp. finitely-generated U_ζ-modules) killed by some power of the maximal ideal of $\mathbb{C}[\Lambda]$ (resp. $Z_{Har}(U_\zeta)$) corresponding to $t \in H$.

THEOREM **2.8.** *Let $\ell > h_G$, and assume that Conjecture 2.6 is valid. If $t \in H$ is regular, then the natural functors*

$$R\Gamma_{\hat{t}} : D^b(\operatorname{Mod}_{coh,t}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})) \to D^b(\operatorname{Mod}_{f,t}(U_\zeta)),$$
$$R\Gamma_t : D^b(\operatorname{Mod}_{coh}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})) \to D^b(\operatorname{Mod}_f(U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t))$$

give equivalences of derived categories.

The proof of this result is completely similar to that of the corresponding fact for Lie algebras in positive characteristics given in [4]. We only give below an outline of it. First note the following.

PROPOSITION **2.9** (Brown-Goodearl [6]). *U_ζ has finite homological dimension.*

The functors

$$R\Gamma_{\hat{t}} : D^b(\mathrm{Mod}_{coh,t}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta})) \to D^b(\mathrm{Mod}_{f,t}(U_\zeta)),$$
$$R\Gamma_t : D^-(\mathrm{Mod}_{coh}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})) \to D^-(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t))$$

have left adjoints

$$\mathcal{L}_{\hat{t}} : D^b(\mathrm{Mod}_{f,t}(U_\zeta)) \to D^b(\mathrm{Mod}_{coh,t}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta})),$$
$$\mathcal{L}_t : D^-(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t)) \to D^-(\mathrm{Mod}_{coh}(Fr_*\mathcal{D}_{\mathcal{B}_\zeta,t})).$$

Arguing exactly as in [4, 3.3, 3.4] using Theorem 2.4 and Proposition 2.9 we obtain the following.

PROPOSITION **2.10.** (i) *If t is regular, the adjunction morphism* $\mathrm{Id} \to R\Gamma_{\hat{t}} \circ \mathcal{L}_{\hat{t}}$ *is an isomorphism on* $D^b(\mathrm{Mod}_{f,t}(U_\zeta))$.
(ii) *For any t, the adjunction morphism* $\mathrm{Id} \to R\Gamma_t \circ \mathcal{L}_t$ *is an isomorphism on* $D^-(\mathrm{Mod}_f(U_\zeta \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}_t))$.

Arguing exactly as in [4, 3.5] using Theorem 2.4, Proposition 2.10 and Lemma 2.11 below we obtain Theorem 2.8. Details are omitted.

LEMMA **2.11** ([21]). *The variety $\mathcal{V}$ is a symplectic manifold.*

2.5. **Finite part.**

2.5.1. In [20] we also introduced a quotient algebra D'_ζ of E_ζ, which is closely related to D_ζ. Let us recall its definition. Take bases $\{x_p\}_p$, $\{y_p\}_p$, $\{x^L_p\}_p$, $\{y^L_p\}_p$ of U^+_ζ, U^-_ζ, $U^{L,+}_\zeta$, $U^{L,-}_\zeta$ respectively such that

$$\tau^L_\zeta(x_{p_1}, y^L_{p_2}) = \delta_{p_1,p_2}, \qquad {}^L\tau_\zeta(x^L_{p_1}, y_{p_2}) = \delta_{p_1,p_2}.$$

We assume that

$$x_p \in U^+_{\zeta,\beta_p}, \quad y_p \in U^-_{\zeta,-\beta_p}, \quad x^L_p \in U^{L,+}_{\zeta,\beta_p}, \quad y^L_p \in U^{L,-}_{\zeta,-\beta_p}$$

for $\beta_p \in Q^+$.

For $\varphi \in A_\zeta(\lambda)_\xi$ with $\lambda \in \Lambda^+$, $\xi \in \Lambda$ we set

$$\Omega'_1(\varphi) = \sum_p (y^L_p \cdot \varphi)x_p \in E_{\zeta,\diamondsuit},$$
$$\Omega'_2(\varphi) = \sum_p ((Sx^L_p) \cdot \varphi)y_p k_{\beta_p} k_{2\xi} e(-2\lambda) \in E_{\zeta,\diamondsuit},$$
$$\Omega'(\varphi) = \Omega'_1(\varphi) - \Omega'_2(\varphi) \in E_{\zeta,\diamondsuit}.$$

We extend Ω' to whole A_ζ by linearity. Then D'_ζ is defined by

$$D'_\zeta = E_\zeta / \sum_{\varphi \in A_\zeta} A_\zeta \Omega'(\varphi) U_\zeta \mathbb{C}[\Lambda].$$

We have a sequence

$$E_\zeta \to D'_\zeta \to D_\zeta$$

of surjective homomorphisms of Λ-graded algebras. Moreover, $D'_\zeta \to D_\zeta$ induces an isomorphism

$$\omega^* D'_\zeta \cong \omega^* D_\zeta \tag{2.7}$$

in $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ (see [20, Corollary 6.6]).

2.5.2. We set

$$U^0_{\mathbb{F},\diamondsuit} = \bigoplus_{\lambda \in \Lambda} \mathbb{F} k_{2\lambda} \subset U^0_{\mathbb{F}}, \qquad U_{\mathbb{F},\diamondsuit} = S(U^-_{\mathbb{F}}) U^0_{\mathbb{F},\diamondsuit} U^+_{\mathbb{F}} \subset U_{\mathbb{F}}.$$

Then we see easily the following.

Lemma **2.12.** *$U_{\mathbb{F},\diamondsuit}$ is an $\mathrm{ad}(U_{\mathbb{F}})$-stable subalgebra of $U_{\mathbb{F}}$.*

Set

$$U_{\mathbb{F},f} = \{u \in U_{\mathbb{F}} \mid \dim \mathrm{ad}(U_{\mathbb{F}})(u) < \infty\}. \tag{2.8}$$

Then $U_{\mathbb{F},f}$ is a subalgebra of $U_{\mathbb{F}}$. Moreover, by Joseph-Letzter [12] we have

$$U_{\mathbb{F},f} = \sum_{\lambda \in \Lambda^+} \mathrm{ad}(U_{\mathbb{F}})(k_{-2\lambda}), \tag{2.9}$$

and hence $U_{\mathbb{F},f}$ is a subalgebra of $U_{\mathbb{F},\diamondsuit}$. $U_{\mathbb{F},\diamondsuit}$ and $U_{\mathbb{F},f}$ are not Hopf subalgebras of $U_{\mathbb{F}}$; nevertheless, they satisfy the following.

Lemma **2.13.** *We have*

$$\Delta(U_{\mathbb{F},f}) \subset U_{\mathbb{F}} \otimes U_{\mathbb{F},f}, \qquad \Delta(U_{\mathbb{F},\diamondsuit}) \subset U_{\mathbb{F}} \otimes U_{\mathbb{F},\diamondsuit}.$$

Proof. For $u \in U_{\mathbb{F}}$ and $\lambda \in \Lambda^+$ we have

$$\begin{aligned} \Delta(\mathrm{ad}(u)(k_{-2\lambda})) &= \sum_{(u)} \Delta(u_{(0)} k_{-2\lambda} (S u_{(1)}) \\ &= \sum_{(u)_3} u_{(0)} k_{-2\lambda} (S u_{(3)}) \otimes u_{(1)} k_{-2\lambda} (S u_{(2)}) \\ &= \sum_{(u)_2} u_{(0)} k_{-2\lambda} (S u_{(2)}) \otimes \mathrm{ad}(u_{(1)})(k_{-2\lambda}). \end{aligned}$$

Hence the first formula follows from (2.9). Since $U_{\mathbb{F},\diamondsuit}$ is generated by e_i, Sf_i for $i \in I$ and $k_{2\lambda}$ for $\lambda \in \Lambda$, the second formula is a consequence of the fact that $\Delta(e_i)$, $\Delta(Sf_i)$, $\Delta(k_{2\lambda})$ belong to $U_{\mathbb{F}} \otimes U_{\mathbb{F},\diamondsuit}$. □

We set

$$E_{\mathbb{F},\diamondsuit} = A_{\mathbb{F}} \otimes U_{\mathbb{F},\diamondsuit} \otimes \mathbb{F}[\Lambda] \subset E_{\mathbb{F}},$$
$$E_{\mathbb{F},f} = A_{\mathbb{F}} \otimes U_{\mathbb{F},f} \otimes \mathbb{F}[\Lambda] \subset E_{\mathbb{F}}.$$

By Lemma 2.13 they are subalgebras of $E_{\mathbb{F}}$.

We set

$$U^0_{\mathbb{A},\diamondsuit} = U^0_{\mathbb{F},\diamondsuit} \cap U_{\mathbb{A}} = \bigoplus_{\lambda\in\Lambda} \mathbb{A}k_{2\lambda}, \quad U_{\mathbb{A},\diamondsuit} = U_{\mathbb{F},\diamondsuit} \cap U_{\mathbb{A}} = S(U^-_{\mathbb{A}})U^0_{\mathbb{A},\diamondsuit}U^+_{\mathbb{A}},$$
$$U_{\mathbb{A},f} = U_{\mathbb{A}} \cap U_{\mathbb{F},f},$$

and

$$E_{\mathbb{A},\diamondsuit} = E_{\mathbb{A}} \cap E_{\mathbb{F},\diamondsuit} = A_{\mathbb{A}} \otimes U_{\mathbb{A},\diamondsuit} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{F},\diamondsuit},$$
$$E_{\mathbb{A},f} = E_{\mathbb{A}} \cap E_{\mathbb{F},f} = A_{\mathbb{A}} \otimes U_{\mathbb{A},f} \otimes \mathbb{A}[\Lambda] \subset E_{\mathbb{F},f},$$

We also set

$$E_{\zeta,\diamondsuit} = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A},\diamondsuit} = A_{\zeta} \otimes U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda] \subset E_{\zeta},$$
$$E_{\zeta,f} = \mathbb{C} \otimes_{\mathbb{A}} E_{\mathbb{A},f} = A_{\zeta} \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \subset E_{\zeta},$$

and

$$D_{\zeta,\diamondsuit} = \mathrm{Im}(E_{\zeta,\diamondsuit} \to D_{\zeta}), \qquad D_{\zeta,f} = \mathrm{Im}(E_{\zeta,f} \to D_{\zeta}),$$
$$D'_{\zeta,\diamondsuit} = \mathrm{Im}(E_{\zeta,\diamondsuit} \to D'_{\zeta}), \qquad D'_{\zeta,f} = \mathrm{Im}(E_{\zeta,f} \to D'_{\zeta}).$$

By

$$E_{\zeta} \cong E_{\zeta,\diamondsuit} \otimes_{U_{\zeta,\diamondsuit}} U_{\zeta}$$

we obtain

$$D'_{\zeta,\diamondsuit} = E_{\zeta,\diamondsuit} / \sum_{\varphi\in A_{\zeta}} A_{\zeta}\Omega'(\varphi)U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda], \tag{2.10}$$

$$D'_{\zeta} \cong D'_{\zeta,\diamondsuit} \otimes_{U_{\zeta,\diamondsuit}} U_{\zeta}. \tag{2.11}$$

2.5.3. Since U_{ζ} is a free $U_{\zeta,\diamondsuit}$-module, we have

$$R^i\Gamma(\omega^* D'_{\zeta}) \cong R^i\Gamma(\omega^* D'_{\zeta,\diamondsuit}) \otimes_{U_{\zeta,\diamondsuit}} U_{\zeta}$$

for any $i \in \mathbb{Z}$. Since $U_{\zeta,\diamondsuit}$ is a localization of $U_{\zeta,f}$ with respect to the Ore subset $\{k_{-2\lambda} \mid \Lambda \in \Lambda^+\}$, we have

$$R^i\Gamma(\omega^* D'_{\zeta,\diamondsuit}) \cong R^i\Gamma(\omega^* D'_{\zeta,f}) \otimes_{U_{\zeta,f}} U_{\zeta,\diamondsuit}$$

for any $i \in \mathbb{Z}$. It follows that

$$R^i\Gamma(\omega^* D'_{\zeta}) \cong R^i\Gamma(\omega^* D'_{\zeta,f}) \otimes_{U_{\zeta,f}} U_{\zeta} \tag{2.12}$$

for any $i \in \mathbb{Z}$. Note

$$R^i\Gamma(\mathcal{B}, Fr_*\mathcal{D}_{\mathcal{B}_\zeta}) \cong R^i\Gamma(\omega^* D'_{\zeta})$$

by Lemma 1.8 and (2.7). Hence Conjecture 2.6 is a consequence of the following stronger conjecture.

Conjecture **2.14.** *Assume $\ell > h_G$. We have*

$$\Gamma(\omega^* D'_{\zeta,f}) \cong U_{\zeta,f} \otimes_{Z_{Har}(U_\zeta)} \mathbb{C}[\Lambda],$$

and

$$R^i\Gamma(\omega^* D'_{\zeta,f}) = 0$$

for $i \neq 0$.

In the rest of this paper we give a reformulation of Conjecture 2.14 in terms of the induction functor.

3. Representations

3.1. For simplicity we introduce a new notation $\tilde{U}_{\mathbb{F}}^- = S(U_{\mathbb{F}}^-)$. Then we have $\tilde{U}_{\mathbb{F}}^- = \langle \tilde{f}_i \mid i \in I\rangle$, where $\tilde{f}_i = f_i k_i$ for $i \in I$. Moreover, setting

$$\tilde{U}_{\mathbb{F},\gamma}^- = \{u \in \tilde{U}_{\mathbb{F}}^- \mid k_\mu u k_{-\mu} = q^{(\gamma,\mu)} u \quad (\mu \in \Lambda)\}$$

for $\gamma \in Q$ we have

$$\tilde{U}_{\mathbb{F}}^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{\mathbb{F},-\gamma}^-, \qquad \tilde{U}_{\mathbb{F},-\gamma}^- = U_{\mathbb{F},-\gamma}^- k_\gamma \quad (\gamma \in Q^+).$$

We also set

$$\tilde{U}_{\mathbb{A}} = U_{\mathbb{A}} \cap \tilde{U}_{\mathbb{F}}, \qquad \tilde{U}_{\mathbb{A},-\gamma} = U_{\mathbb{A}} \cap \tilde{U}_{\mathbb{F},-\gamma} \quad (\gamma \in Q^+),$$
$$\tilde{U}_{\zeta} = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}_{\mathbb{A}}, \qquad \tilde{U}_{\zeta,-\gamma} = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}_{\mathbb{A},-\gamma} \quad (\gamma \in Q^+).$$

Then we have

$$\tilde{U}_{\mathbb{A}}^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{\mathbb{A},-\gamma}^-, \qquad \tilde{U}_{\zeta}^- = \bigoplus_{\gamma \in Q^+} \tilde{U}_{\zeta,-\gamma}^-.$$

3.2. For $\lambda \in \Lambda$ we define an algebra homomorphism $\chi_\lambda : U_{\mathbb{F}}^0 \to \mathbb{F}$ by $\chi_\lambda(k_\mu) = q^{(\lambda,\mu)}$ $(\mu \in \Lambda)$. For $M \in \mathrm{Mod}(U_{\mathbb{F}})$ and $\lambda \in \Lambda$ we set

$$M_\lambda = \{m \in M \mid hm = \chi_\lambda(h) m \quad (h \in U_{\mathbb{F}}^0)\}.$$

For $\lambda \in \Lambda$ we define $M_{+,\mathbb{F}}(\lambda), M_{-,\mathbb{F}}(\lambda) \in \mathrm{Mod}(U_{\mathbb{F}})$ by

$$M_{+,\mathbb{F}}(\lambda) = U_{\mathbb{F}} / \sum_{y \in U_{\mathbb{F}}^-} U_{\mathbb{F}}(y - \varepsilon(y)) + \sum_{h \in U_{\mathbb{F}}^0} U_{\mathbb{F}}(h - \chi_\lambda(h)),$$
$$M_{-,\mathbb{F}}(\lambda) = U_{\mathbb{F}} / \sum_{x \in U_{\mathbb{F}}^+} U_{\mathbb{F}}(x - \varepsilon(x)) + \sum_{h \in U_{\mathbb{F}}^0} U_{\mathbb{F}}(h - \chi_\lambda(h)).$$

$M_{+,\mathbb{F}}(\lambda)$ is a lowest weight module with lowest weight λ, and $M_{-,\mathbb{F}}(\lambda)$ is a highest weight module with highest weight λ. We have isomorphisms

$$M_{+,\mathbb{F}}(\lambda) \cong U_{\mathbb{F}}^+ \quad (\overline{u} \leftrightarrow u), \qquad M_{-,\mathbb{F}}(\lambda) \cong U_{\mathbb{F}}^- \quad (\overline{u} \leftrightarrow u)$$

of $\mathbb{F}$-modules. Moreover, we have weight space decompositions

$$M_{+,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda + Q^+} M_{+,\mathbb{F}}(\lambda)_\mu, \qquad M_{-,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} M_{-,\mathbb{F}}(\lambda)_\mu.$$

For $\lambda \in \Lambda^+$ we define $L_{+,\mathbb{F}}(-\lambda), L_{-,\mathbb{F}}(\lambda) \in \mathrm{Mod}_f(U_\mathbb{F})$ by

$$\begin{aligned} L_{+,\mathbb{F}}(-\lambda) =& U_\mathbb{F} / \sum_{y \in U_\mathbb{F}^-} U_\mathbb{F}(y - \varepsilon(y)) + \sum_{h \in U_\mathbb{F}^0} U_\mathbb{F}(h - \chi_{-\lambda}(h)) + \sum_{i \in I} U_\mathbb{F} e_i^{((\lambda,\alpha_i^\vee)+1)}, \\ L_{-,\mathbb{F}}(\lambda) =& U_\mathbb{F} / \sum_{x \in U_\mathbb{F}^+} U_\mathbb{F}(x - \varepsilon(x)) + \sum_{h \in U_\mathbb{F}^0} U_\mathbb{F}(h - \chi_{\lambda}(h)) + \sum_{i \in I} U_\mathbb{F} f_i^{((\lambda,\alpha_i^\vee)+1)}. \end{aligned}$$

$L_{+,\mathbb{F}}(-\lambda)$ is a finite-dimensional irreducible lowest weight module with lowest weight $-\lambda$, and $L_{-,\mathbb{F}}(\lambda)$ is a finite-dimensional irreducible highest weight module with highest weight λ. We have weight space decompositions

$$L_{+,\mathbb{F}}(-\lambda) = \bigoplus_{\mu \in -\lambda + Q^+} L_{+,\mathbb{F}}(-\lambda)_\mu, \qquad L_{-,\mathbb{F}}(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} L_{-,\mathbb{F}}(\lambda)_\mu.$$

For $\lambda \in \Lambda^+$ we have isomorphisms

$$\begin{aligned} L_{+,\mathbb{F}}(-\lambda) &\cong U_\mathbb{F}^{L,+} / \sum_{i \in I} U_\mathbb{F}^{L,+} e_i^{((\lambda,\alpha_i^\vee)+1)}, \quad (\overline{u} \leftrightarrow \overline{u}), \\ L_{-,\mathbb{F}}(\lambda) &\cong \tilde{U}_\mathbb{F}^{L,-} / \sum_{i \in I} \tilde{U}_\mathbb{F}^{L,-} \tilde{f}_i^{((\lambda,\alpha_i^\vee)+1)}, \quad (\overline{u} \leftrightarrow \overline{u}) \end{aligned}$$

of vector spaces (Lusztig [13]).

Let M be a $U_\mathbb{F}$-module with weight space decomposition $M = \bigoplus_{\mu \in \Lambda} M_\mu$ such that $\dim M_\mu < \infty$ for any $\mu \in \Lambda$. We define a $U_\mathbb{F}$-module $M^\bigstar$ by

$$M^\bigstar = \bigoplus_{\mu \in \Lambda} M_\mu^* \subset M^* = \mathrm{Hom}_\mathbb{F}(M, \mathbb{F}),$$

where the action of $U_\mathbb{F}$ is given by

$$\langle um^*, m \rangle = \langle m^*, (Su)m \rangle \qquad (u \in U_\mathbb{F}, m^* \in M^\bigstar, m \in M).$$

Here $\langle\, , \,\rangle : M^\bigstar \times M \to \mathbb{F}$ is the natural paring.

We set

$$\begin{aligned} M_{\pm,\mathbb{F}}^*(\lambda) &= (M_{\mp,\mathbb{F}}(-\lambda))^\bigstar \qquad (\lambda \in \Lambda), \\ L_{\pm,\mathbb{F}}^*(\mp\lambda) &= (L_{\mp,\mathbb{F}}(\pm\lambda))^\bigstar \qquad (\lambda \in \Lambda^+). \end{aligned}$$

Since $L_{\mp,\mathbb{F}}(\pm\lambda)$ is irreducible we have

$$L_{\pm,\mathbb{F}}^*(\mp\lambda) \cong L_{\pm,\mathbb{F}}(\mp\lambda) \qquad (\lambda \in \Lambda^+).$$

We define isomorphisms

$$\Phi_\lambda : U_\mathbb{F}^+ \to M_{+,\mathbb{F}}^*(\lambda), \qquad \Psi_\lambda : \tilde{U}_\mathbb{F}^- \to M_{-,\mathbb{F}}^*(\lambda) \tag{3.1}$$

of vector spaces by

$$\begin{aligned} \langle \Phi_\lambda(x), \overline{v} \rangle &= \tau(x, v) \qquad (x \in U_\mathbb{F}^+, v \in \tilde{U}_\mathbb{F}^-), \\ \langle \Psi_\lambda(y), \overline{Su} \rangle &= \tau(u, y) \qquad (y \in \tilde{U}_\mathbb{F}^-, u \in U_\mathbb{F}^+). \end{aligned}$$

LEMMA **3.1.** (i) *The $U_{\mathbb{F}}$-module structure of $M^*_{+,\mathbb{F}}(\lambda)$ is given by*

$$h \cdot \Phi_\lambda(x) = \chi_{\lambda+\gamma}(h)\Phi_\lambda(x) \quad (x \in U^+_{\mathbb{F},\gamma}, h \in U^0_{\mathbb{F}}), \tag{3.2}$$

$$v \cdot \Phi_\lambda(x) = \sum_{(x)} \tau(x_{(0)}, Sv)\Phi_\lambda(x_{(1)}) \quad (x \in U^+_{\mathbb{F}}, v \in U^-_{\mathbb{F}}), \tag{3.3}$$

$$u \cdot \Phi_\lambda(x) = \Phi_\lambda(k_{-\lambda}(\mathrm{ad}(u)(k_\lambda x k_\lambda))k_{-\lambda}) \quad (x \in U^+_{\mathbb{F}},\ u \in U^+_{\mathbb{F}}). \tag{3.4}$$

(ii) *The $U_{\mathbb{F}}$-module structure of $M^*_{-,\mathbb{F}}(\lambda)$ is given by*

$$h \cdot \Psi_\lambda(y) = \chi_{\lambda-\gamma}(h)\Psi_\lambda(y) \quad (y \in \tilde{U}^-_{\mathbb{F},-\gamma}, h \in U^0_{\mathbb{F}}), \tag{3.5}$$

$$u \cdot \Psi_\lambda(y) = \sum_{(y)} \tau(u, y_{(0)})\Psi_\lambda(y_{(1)}) \quad (y \in \tilde{U}^-_{\mathbb{F}}, u \in U^+_{\mathbb{F}}), \tag{3.6}$$

$$v \cdot \Psi_\lambda(y) = \Psi_\lambda(k_\lambda(\mathrm{ad}(v)(k_{-\lambda} y k_{-\lambda}))k_\lambda) \quad (y \in \tilde{U}^-_{\mathbb{F}},\ v \in U^-_{\mathbb{F}}). \tag{3.7}$$

PROOF. We will only prove (i). The proof of (ii) is similar and omitted. Note that for $x \in U^+_{\mathbb{F}}$, $a \in U_{\mathbb{F}}$, $v \in \tilde{U}^-_{\mathbb{F}}$ we have

$$\langle a \cdot \Phi_\lambda(x), \overline{v} \rangle = \langle \Phi_\lambda(x), \overline{(Sa)v} \rangle.$$

Let us show (3.2). For $v \in \tilde{U}^-_{\mathbb{F},-\delta}$ we have

$$\begin{aligned} &\langle h \cdot \Phi_\lambda(x), \overline{v} \rangle = \langle \Phi_\lambda(x), \overline{(Sh)v} \rangle = \delta_{\gamma,\delta}\langle \Phi_\lambda(x), \overline{(Sh)v} \rangle \\ =&\delta_{\gamma,\delta}\chi_{\lambda+\gamma}(h)\langle \Phi_\lambda(x), \overline{v} \rangle = \chi_{\lambda+\gamma}(h)\langle \Phi_\lambda(x), \overline{v} \rangle. \end{aligned}$$

Hence(3.2) holds. Let us next show (3.3). For $v \in \tilde{U}^-_{\mathbb{F}}$ we have

$$\langle y \cdot \Phi_\lambda(x), \overline{v} \rangle = \langle \Phi_\lambda(x), \overline{(Sy)v} \rangle = \tau(x, (Sy)v) = \sum_{(x)} \tau(x_{(0)}, Sy)\tau(x_{(1)}, v)$$

$$= \left\langle \Phi_\lambda \left(\sum_{(x)} \tau(x_{(0)}, Sy)x_{(1)} \right), \overline{v} \right\rangle$$

Hence (3.3) also holds. Let us finally show (3.4). We may assume that $u \in U^+_{\mathbb{F},\beta}$ for some $\beta \in Q^+$. Then we can write

$$\Delta u = \sum_j u_j k_{\beta'_j} \otimes u'_j \quad (\beta_j, \beta'_j \in Q^+, \beta_j + \beta'_j = \beta,\ u_j \in U^+_{\mathbb{F},\beta_j},\ u'_j \in U^+_{\mathbb{F},\beta'_j}).$$

For $v \in \tilde{U}_{\mathbb{F}}^-$ we have

$$
\begin{aligned}
&\langle u \cdot \Phi_\lambda(x), \overline{v} \rangle = \langle \Phi_\lambda(x), \overline{(Su)v} \rangle \\
=&\sum_{(u)_2,(v)_2} \tau(Su_{(2)}, v_{(0)})\tau(Su_{(0)}, Sv_{(2)})\langle \Phi_\lambda(x), \overline{v_{(1)}(Su_{(1)})} \rangle \\
=&\sum_{j,(v)_2} \tau(Su'_j, v_{(0)})\tau(u_j k_{\beta'_j}, v_{(2)})\langle \Phi_\lambda(x), \overline{v_{(1)}(Sk_{\beta'_j})} \rangle \\
=&\sum_{j,(v)_2} q^{(\lambda,\beta'_j-\beta_j)}\tau(Su'_j, v_{(0)})\tau(u_j k_{\beta'_j}, v_{(2)})\langle \Phi_\lambda(x), \overline{v_{(1)}k_{-\beta_j}} \rangle \\
=&\sum_{j,(v)_2} q^{(\lambda,\beta'_j-\beta_j)}\tau(Su'_j, v_{(0)})\tau(u_j k_{\beta'_j}, v_{(2)})\tau(x, v_{(1)}k_{-\beta_j}) \\
=&\sum_{j,(v)_2} q^{(\lambda,\beta'_j-\beta_j)}\tau(Su'_j, v_{(0)})\tau(x, v_{(1)})\tau(u_j k_{\beta'_j}, v_{(2)}) \\
=&\sum_{j} q^{(\lambda,\beta'_j-\beta_j)}\tau(u_j k_{\beta'_j} x(Su'_j), v) \\
=&\langle \Phi_\lambda(k_{-\lambda}(\mathrm{ad}(u)(k_\lambda x k_\lambda))k_{-\lambda}), \overline{v} \rangle.
\end{aligned}
$$

Here, we have used Lemma 1.1. Note also that $\Delta\tilde{U}_{\mathbb{F}}^- \subset \sum_{\gamma\in Q^+} \tilde{U}_{\mathbb{F}}^- k_\gamma \otimes \tilde{U}_{\mathbb{F},-\gamma}^-$, and hence $\Delta_2\tilde{U}_{\mathbb{F}}^- \subset \sum_{\gamma,\delta\in Q^+} \tilde{U}_{\mathbb{F}}^- k_{\gamma+\delta} \otimes \tilde{U}_{\mathbb{F},-\gamma}^- k_\delta \otimes \tilde{U}_{\mathbb{F},-\delta}^-$. (3.4) is proved. $\square$

For $\lambda \in \Lambda$ we denote by $\mathbb{F}_\lambda^{\geqq 0} = \mathbb{F}1_\lambda^{\geqq 0}$ (resp. $\mathbb{F}_\lambda^{\leqq 0} = \mathbb{F}1_\lambda^{\leqq 0}$) the one-dimensional $U_{\mathbb{F}}^{\geqq 0}$-module (resp. $U_{\mathbb{F}}^{\leqq 0}$-module) such that $h1_\lambda^{\geqq 0} = \chi_\lambda(h)1_\lambda^{\geqq 0}$, $u1_\lambda^{\geqq 0} = \varepsilon(u)1_\lambda^{\geqq 0}$ for $h \in U_{\mathbb{F}}^0$ and $u \in U_{\mathbb{F}}^+$ (resp. $h1_\lambda^{\leqq 0} = \chi_\lambda(h)1_\lambda^{\leqq 0}$, $u1_\lambda^{\leqq 0} = \varepsilon(u)1_\lambda^{\leqq 0}$ for $h \in U_{\mathbb{F}}^0$ and $u \in U_{\mathbb{F}}^-$).

Note that for any $\lambda \in \Lambda$, $k_{-2\lambda}U_{\mathbb{F}}^+$ (resp. $\tilde{U}_{\mathbb{F}}^- k_{-2\lambda}$) is $\mathrm{ad}(U_{\mathbb{F}}^{\geqq 0})$-stable (resp. $\mathrm{ad}(U_{\mathbb{F}}^{\leqq 0})$-stable). We see easily from Lemma 3.1 the following.

Lemma **3.2.** *Let* $\lambda \in \Lambda$.

(i) *The linear map*

$$k_{-2\lambda}U_{\mathbb{F}}^+ \to M_{+,\mathbb{F}}^*(-\lambda) \otimes \mathbb{F}_\lambda^{\geqq 0} \qquad (k_{-\lambda} x k_{-\lambda} \mapsto \Phi_{-\lambda}(x) \otimes 1_\lambda^{\geqq 0})$$

is an isomorphism of $U_{\mathbb{F}}^{\geqq 0}$*-modules, where* $k_{-2\lambda}U_{\mathbb{F}}^+$ *is regarded as a* $U_{\mathbb{F}}^{\leqq 0}$*-module by the adjoint action.*

(ii) *The linear map*

$$\tilde{U}_{\mathbb{F}}^- k_{-2\lambda} \to \mathbb{F}_{-\lambda}^{\leqq 0} \otimes M_{-,\mathbb{F}}^*(\lambda) \qquad (k_{-\lambda} y k_{-\lambda} \mapsto 1_{-\lambda}^{\leqq 0} \otimes \Psi_\lambda(y))$$

is an isomorphism of $U_{\mathbb{F}}^{\leqq 0}$*-modules, where* $\tilde{U}_{\mathbb{F}}^- k_{-2\lambda}$ *is regarded as a* $U_{\mathbb{F}}^{\leqq 0}$*-module by the adjoint action.*

We have an injective $U_{\mathbb{F}}$-homomorphism

$$L_{\pm,\mathbb{F}}^*(\mp\lambda) \to M_{\pm,\mathbb{F}}^*(\mp\lambda) \qquad (\lambda \in \Lambda^+). \tag{3.8}$$

induced by the natural homomorphism $M_{\pm,\mathbb{F}}(\mp\lambda) \to L_{\pm,\mathbb{F}}(\mp\lambda)$. For $\lambda \in \Lambda^+$ we define subspaces $U^+_{\mathbb{F}}(\lambda)$, $\tilde{U}^-_{\mathbb{F}}(\lambda)$ of $U^+_{\mathbb{F}}$, $\tilde{U}^-_{\mathbb{F}}$ respectively by

$$U^+_{\mathbb{F}}(\lambda) = \Phi^{-1}_{-\lambda}(L^*_{+,\mathbb{F}}(-\lambda)), \qquad \tilde{U}^-_{\mathbb{F}}(\lambda) = \Psi^{-1}_{\lambda}(L^*_{-,\mathbb{F}}(\lambda)).$$

Lemma **3.3.** (i) *For $\lambda, \mu \in \Lambda^+$ we have*

$$U^+_{\mathbb{F}}(\lambda) \subset U^+_{\mathbb{F}}(\lambda+\mu), \qquad \tilde{U}^-_{\mathbb{F}}(\lambda) \subset \tilde{U}^-_{\mathbb{F}}(\lambda+\mu).$$

(ii) *We have*

$$U^+_{\mathbb{F}} = \sum_{\lambda\in\Lambda^+} U^+_{\mathbb{F}}(\lambda), \qquad \tilde{U}^-_{\mathbb{F}} = \sum_{\lambda\in\Lambda^+} \tilde{U}^-_{\mathbb{F}}(\lambda).$$

Proof. We will only prove the statements for $U^+_{\mathbb{F}}$. By definition we have $U^+_{\mathbb{F}}(\lambda) = \{x \in U^+_{\mathbb{F}} \mid \tau(x, I_\lambda) = \{0\}\}$, where $I_\lambda = \sum_{i\in I} \tilde{U}^-_{\mathbb{F}} \tilde{f}_i^{((\lambda,\alpha_i^\vee)+1)}$. Hence (i) is a consequence of $I_\lambda \supset I_{\lambda+\mu}$ for $\lambda, \mu \in \Lambda^+$. To show (ii) it is sufficient to show that for any $\beta \in Q^+$ there exists some $\lambda \in \Lambda^+$ such that $U^+_{\mathbb{F},\beta} \subset U^+_{\mathbb{F}}(\lambda)$. Set $m = \mathrm{ht}(\beta)$. If $\lambda \in \Lambda^+$ satisfies $(\lambda, \alpha_i^\vee) \geqq m$ for any $i \in I$, then we have $I_\lambda \subset \bigoplus_{\gamma\in Q^+,\mathrm{ht}(\gamma)>m} \tilde{U}^-_{\mathbb{F},-\gamma}$. From this we obtain $\tau(U^+_{\mathbb{F},\beta}, I_\lambda) = \{0\}$, and hence $U^+_{\mathbb{F},\beta} \subset U^+_{\mathbb{F}}(\lambda)$. □

Lemma **3.4.** *For $\lambda \in \Lambda^+$ we have*

$$\tilde{U}^-_{\mathbb{F}}(\lambda)k_{-2\lambda} \subset U_{\mathbb{F},f}, \qquad k_{-2\lambda}U^+_{\mathbb{F}}(\lambda) \subset U_{\mathbb{F},f}$$

Proof. By Lemma 3.2 we have an isomorphism

$$k_{-2\lambda}U^+_{\mathbb{F}}(\lambda) \to L^*_{+,\mathbb{F}}(-\lambda) \otimes \mathbb{F}^{\geqq 0}_{\lambda} \qquad (k_{-\lambda}xk_{-\lambda} \mapsto \Phi_{-\lambda}(x) \otimes 1^{\geqq 0}_{\lambda})$$

of $U^{\geqq 0}_{\mathbb{F}}$-modules. We have $L^*_{+,\mathbb{F}}(-\lambda) \cong L_{+,\mathbb{F}}(-\lambda)$ and hence $L^*_{+,\mathbb{F}}(-\lambda)\otimes \mathbb{F}^{\geqq 0}_{\lambda}$ is generated by $\Phi_{-\lambda}(1) \otimes 1^{\geqq 0}_{\lambda}$ as a $U^{\geqq 0}_{\mathbb{F}}$-module. It follows that

$$k_{-2\lambda}U^+_{\mathbb{F}}(\lambda) = \mathrm{ad}(U^{\geqq 0}_{\mathbb{F}})(k_{-2\lambda}) \subset U_{\mathbb{F},f}$$

by (2.9). The proof of $\tilde{U}^-_{\mathbb{F}}(\lambda)k_{-2\lambda} \subset U_{\mathbb{F},f}$ is similar. □

3.3. It is well-known that for $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$ there exists $h \in U^{L,0}_{\mathbb{A}}$ such that $\chi_\lambda(h) = 1$ and $\chi_\mu(h) = 0$. In particular, we have $\chi_\lambda \neq \chi_\mu$ (see for example [20, Lemma 2.3]).

For $M \in \mathrm{Mod}(U^L_{\mathbb{A}})$ and $\lambda \in \Lambda$ we set

$$M_\lambda = \{m \in M \mid hm = \chi_\lambda(h)m \quad (h \in U^{L,0}_{\mathbb{A}})\}.$$

For $\lambda \in \Lambda$ we define $M_{+,\mathbb{A}}(\lambda), M_{-,\mathbb{A}}(\lambda) \in \mathrm{Mod}(U^L_{\mathbb{A}})$ by

$$M_{+,\mathbb{A}}(\lambda) = U^L_{\mathbb{A}}/\sum_{y\in U^{L,-}_{\mathbb{A}}} U^L_{\mathbb{A}}(y-\varepsilon(y)) + \sum_{h\in U^{L,0}_{\mathbb{A}}} U^L_{\mathbb{A}}(h-\chi_\lambda(h)),$$

$$M_{-,\mathbb{A}}(\lambda) = U^L_{\mathbb{A}}/\sum_{x\in U^{L,+}_{\mathbb{A}}} U^L_{\mathbb{A}}(x-\varepsilon(x)) + \sum_{h\in U^{L,0}_{\mathbb{A}}} U^L_{\mathbb{A}}(h-\chi_\lambda(h)).$$

By the triangular decomposition we have isomorphisms

$$M_{+,\mathbb{A}}(\lambda) \cong U^{L,+}_{\mathbb{A}} \quad (\overline{u} \leftrightarrow u), \qquad M_{-,\mathbb{A}}(\lambda) \cong U^{L,-}_{\mathbb{A}} \quad (\overline{u} \leftrightarrow u)$$

of $\mathbb{A}$-modules. In particular, $M_{\pm,\mathbb{A}}(\lambda)$ is a free $\mathbb{A}$-module and we have $\mathbb{F}\otimes_{\mathbb{A}} M_{\pm,\mathbb{A}}(\lambda)\cong M_{\pm,\mathbb{F}}(\lambda)$. Moreover, we have weight space decompositions

$$M_{+,\mathbb{A}}(\lambda)=\bigoplus_{\mu\in\lambda+Q^+} M_{+,\mathbb{A}}(\lambda)_\mu,\qquad M_{-,\mathbb{A}}(\lambda)=\bigoplus_{\mu\in\lambda-Q^+} M_{-,\mathbb{A}}(\lambda)_\mu.$$

For $\lambda\in\Lambda^+$ we define $L_{+,\mathbb{A}}(-\lambda)\in\mathrm{Mod}(U^L_{\mathbb{A}})$ (resp. $L_{-,\mathbb{A}}(\lambda)\in\mathrm{Mod}(U^L_{\mathbb{A}})$) to be the $U^L_{\mathbb{A}}$-submodule of $L_{+,\mathbb{F}}(-\lambda)$ (resp. $L_{-,\mathbb{F}}(\lambda)$) generated by $\overline{1}\in L_{+,\mathbb{F}}(-\lambda)$ (resp. $\overline{1}\in L_{-,\mathbb{F}}(\lambda)$). By definition $L_{\pm,\mathbb{A}}(\mp\lambda)$ is a free $\mathbb{A}$-module and we have $\mathbb{F}\otimes_{\mathbb{A}} L_{\pm,\mathbb{A}}(\mp\lambda)\cong L_{\pm,\mathbb{F}}(\mp\lambda)$. Moreover, we have weight space decompositions

$$L_{+,\mathbb{A}}(-\lambda)=\bigoplus_{\mu\in-\lambda+Q^+} L_{+,\mathbb{A}}(-\lambda)_\mu,\qquad L_{-,\mathbb{A}}(\lambda)=\bigoplus_{\mu\in\lambda-Q^+} L_{-,\mathbb{A}}(\lambda)_\mu.$$

The canonical surjective $U_{\mathbb{F}}$-homomorphism $M_{\pm,\mathbb{F}}(\mp\lambda)\to L_{\pm,\mathbb{F}}(\mp\lambda)$ induces a surjective $U^L_{\mathbb{A}}$-homomorphism

$$M_{\pm,\mathbb{A}}(\mp\lambda)\to L_{\pm,\mathbb{A}}(\mp\lambda)\qquad(\lambda\in\Lambda^+). \tag{3.9}$$

Note that (3.9) a split epimorphism of $\mathbb{A}$-modules since $\mathbb{A}$ is PID, and $M_{\pm,\mathbb{A}}(\mp\lambda)_\mu$, $L_{\pm,\mathbb{A}}(\mp\lambda)_\mu$ are torsion free finitely generated $\mathbb{A}$-modules for each $\mu\in\Lambda$.

Let M be a $U^L_{\mathbb{A}}$-module with weight space decomposition $M=\bigoplus_{\mu\in\Lambda} M_\mu$ such that M_μ is a free $\mathbb{A}$-module of finite rank for any $\mu\in\Lambda$. We define a $U^L_{\mathbb{A}}$-module $M^\bigstar$ by

$$M^\bigstar=\bigoplus_{\mu\in\Lambda}\mathrm{Hom}_{\mathbb{A}}(M_\mu,\mathbb{A})\subset\mathrm{Hom}_{\mathbb{A}}(M,\mathbb{A}),$$

where the action of $U^L_{\mathbb{A}}$ is given by

$$\langle um^*,m\rangle=\langle m^*,(Su)m\rangle\qquad(u\in U^L_{\mathbb{A}},m^*\in M^\bigstar,m\in M).$$

Here $\langle\,,\,\rangle: M^\bigstar\times M\to\mathbb{A}$ is the natural paring.

We set

$$M^*_{\pm,\mathbb{A}}(\lambda)=(M_{\mp,\mathbb{A}}(-\lambda))^\bigstar\qquad(\lambda\in\Lambda),$$
$$L^*_{\pm,\mathbb{A}}(\mp\lambda)=(L_{\mp,\mathbb{A}}(\pm\lambda))^\bigstar\qquad(\lambda\in\Lambda^+).$$

Then $M^*_{\pm,\mathbb{A}}(\lambda)$ for $\lambda\in\Lambda$ and $L^*_{\pm,\mathbb{A}}(\mp\lambda)$ for $\lambda\in\Lambda^+$ are free $\mathbb{A}$-modules satisfying

$$\mathbb{F}\otimes_{\mathbb{A}} M^*_{\pm,\mathbb{A}}(\lambda)\cong M^*_{\pm,\mathbb{F}}(\lambda),\qquad\mathbb{F}\otimes_{\mathbb{A}} L^*_{\pm,\mathbb{A}}(\mp\lambda)\cong L^*_{\pm,\mathbb{F}}(\mp\lambda).$$

Moreover, we can identify $M^*_{\pm,\mathbb{A}}(\lambda)$ and $L^*_{\pm,\mathbb{A}}(\mp\lambda)$ with $\mathbb{A}$-submodules of $M^*_{\pm,\mathbb{F}}(\lambda)$ and $L^*_{\pm,\mathbb{F}}(\mp\lambda)$ respectively. Under this identification we have

$$L^*_{\pm,\mathbb{A}}(\mp\lambda)=L^*_{\pm,\mathbb{F}}(\mp\lambda)\cap M^*_{\pm,\mathbb{A}}(\mp\lambda)\qquad(\lambda\in\Lambda^+). \tag{3.10}$$

In particular, the $U^L_{\mathbb{A}}$-homomorphism

$$L^*_{\pm,\mathbb{A}}(\mp\lambda)\to M^*_{\pm,\mathbb{A}}(\mp\lambda)\qquad(\lambda\in\Lambda^+). \tag{3.11}$$

is a split monomorphism of $\mathbb{A}$-modules.

By abuse of notation we write

$$\Phi_\lambda : U_{\mathbb{A}}^+ \to M_{+,\mathbb{A}}^*(\lambda), \qquad \Psi_\lambda : \tilde{U}_{\mathbb{A}}^- \to M_{-,\mathbb{A}}^*(\lambda) \tag{3.12}$$

the isomorphisms of $\mathbb{A}$-modules induced by (3.1). By Lemma 3.1 we have the following.

LEMMA **3.5.** (i) *The $U_{\mathbb{A}}^L$-module structure of $M_{+,\mathbb{A}}^*(\lambda)$ is given by*

$$h\cdot\Phi_\lambda(x) = \chi_{\lambda+\gamma}(h)\Phi_\lambda(x) \quad (x \in U_{\mathbb{A},\gamma}^+, h \in U_{\mathbb{A}}^{L,0}), \tag{3.13}$$

$$v\cdot\Phi_\lambda(x) = \sum_{(x)} \tau_{\mathbb{A}}^L(x_{(0)}, Sv)\Phi_\lambda(x_{(1)}) \quad (x \in U_{\mathbb{A}}^+, v \in U_{\mathbb{A}}^{L,-}), \tag{3.14}$$

$$u\cdot\Phi_\lambda(x) = \Phi_\lambda(k_{-\lambda}(\mathrm{ad}(u)(k_\lambda x k_\lambda))k_{-\lambda}) \quad (x \in U_{\mathbb{A}}^+,\ u \in U_{\mathbb{A}}^{L,+}). \tag{3.15}$$

(ii) *The $U_{\mathbb{A}}^L$-module structure of $M_{-,\mathbb{A}}^*(\lambda)$ is given by*

$$h\cdot\Psi_\lambda(y) = \chi_{\lambda-\gamma}(h)\Psi_\lambda(y) \quad (y \in \tilde{U}_{\mathbb{A},-\gamma}^-, h \in U_{\mathbb{A}}^{L,0}), \tag{3.16}$$

$$u\cdot\Psi_\lambda(y) = \sum_{(y)} {}^L\tau_{\mathbb{A}}(u, y_{(0)})\Psi_\lambda(y_{(1)}) \quad (y \in \tilde{U}_{\mathbb{A}}^-, u \in U_{\mathbb{A}}^{L,+}), \tag{3.17}$$

$$v\cdot\Psi_\lambda(y) = \Psi_\lambda(k_\lambda(\mathrm{ad}(v)(k_{-\lambda} y k_{-\lambda}))k_\lambda) \quad (y \in \tilde{U}_{\mathbb{A}}^-,\ v \in U_{\mathbb{A}}^{L,-}). \tag{3.18}$$

For $\lambda \in \Lambda^+$ we define $\mathbb{A}$-submodules $U_{\mathbb{A}}^+(\lambda)$, $\tilde{U}_{\mathbb{A}}^-(\lambda)$ of $U_{\mathbb{A}}^+$, $\tilde{U}_{\mathbb{A}}^-$ respectively by

$$U_{\mathbb{A}}^+(\lambda) = \Phi_{-\lambda}^{-1}(L_{+,\mathbb{A}}^*(-\lambda)), \qquad \tilde{U}_{\mathbb{A}}^-(\lambda) = \Psi_\lambda^{-1}(L_{-,\mathbb{A}}^*(\lambda)).$$

The embeddings

$$U_{\mathbb{A}}^+(\lambda) \hookrightarrow U_{\mathbb{A}}^+, \qquad \tilde{U}_{\mathbb{A}}^-(\lambda) \hookrightarrow \tilde{U}_{\mathbb{A}}^- \qquad (\lambda \in \Lambda^+) \tag{3.19}$$

are split monomorphisms of $\mathbb{A}$-modules. By (3.10) we have

$$U_{\mathbb{A}}^+(\lambda) = U_{\mathbb{F}}^+(\lambda) \cap U_{\mathbb{A}}^+, \qquad \tilde{U}_{\mathbb{A}}^-(\lambda) = \tilde{U}_{\mathbb{F}}^-(\lambda) \cap \tilde{U}_{\mathbb{A}}^- \qquad (\lambda \in \Lambda^+). \tag{3.20}$$

In particular, we have

$$U_{\mathbb{A}}^+(\lambda) \subset U_{\mathbb{A}}^+(\lambda+\mu), \quad \tilde{U}_{\mathbb{A}}^-(\lambda) \subset \tilde{U}_{\mathbb{A}}^-(\lambda+\mu) \quad (\lambda, \mu \in \Lambda^+), \tag{3.21}$$

$$U_{\mathbb{A}}^+ = \sum_{\lambda\in\Lambda^+} U_{\mathbb{A}}^+(\lambda), \quad \tilde{U}_{\mathbb{A}}^- = \sum_{\lambda\in\Lambda^+} \tilde{U}_{\mathbb{A}}^-(\lambda), \tag{3.22}$$

$$\tilde{U}_{\mathbb{A}}^-(\lambda)k_{-2\lambda} \subset U_{\mathbb{A},f}, \quad k_{-2\lambda}U_{\mathbb{A}}^+(\lambda) \subset U_{\mathbb{A},f} \qquad (\lambda \in \Lambda^+) \tag{3.23}$$

by Lemma 3.3 and Lemma 3.4.

3.4. Let $\lambda \in \Lambda$. By abuse of notation we also denote by $\chi_\lambda : U_\zeta^{L,0} \to \mathbb{C}$ the $\mathbb{C}$-algebra homomorphism induced by $\chi_\lambda : U_{\mathbb{A}}^{L,0} \to \mathbb{A}$. Then $\{\chi_\lambda\}_{\lambda\in\Lambda}$ is a linearly independent subset of the $\mathbb{C}$-module $\mathrm{Hom}_{\mathbb{C}}(U_\zeta^{L,0}, \mathbb{C})$. For $M \in \mathrm{Mod}(U_\zeta^L)$ and $\lambda \in \Lambda$ we set

$$M_\lambda = \{m \in M \mid hm = \chi_\lambda(h)m\ (h \in U_\zeta^{L,0})\}.$$

For $\lambda \in \Lambda$ we set

$$M_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} M_{\pm,\mathbb{A}}(\lambda), \qquad M^*_{\pm,\zeta}(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} M^*_{\pm,\mathbb{A}}(\lambda).$$

For $\lambda \in \Lambda^+$ we set

$$L_{\pm,\zeta}(\mp\lambda) = \mathbb{C} \otimes_{\mathbb{A}} L_{\pm,\mathbb{A}}(\mp\lambda), \qquad L^*_{\pm,\zeta}(\mp\lambda) = \mathbb{C} \otimes_{\mathbb{A}} L^*_{\pm,\mathbb{A}}(\mp\lambda).$$

We have canonical U^L_ζ-homomorphisms

$$M_{\pm,\zeta}(\mp\lambda) \to L_{\pm,\zeta}(\mp\lambda) \qquad (\lambda \in \Lambda^+), \tag{3.24}$$

$$L^*_{\pm,\zeta}(\mp\lambda) \to M^*_{\pm,\zeta}(\mp\lambda) \qquad (\lambda \in \Lambda^+). \tag{3.25}$$

Note that (3.24) is surjective, and (3.25) is injective.

For any $\lambda \in \Lambda^+$ we have an isomorphism

$$A_\zeta(\lambda) \cong L^*_{-,\zeta}(\lambda) \tag{3.26}$$

of U^L_ζ-modules (see, for example, [20]).

Let $\lambda \in \Lambda$. By abuse of notation we also denote by

$$\Phi_\lambda : U^+_\zeta \to M^*_{+,\zeta}(\lambda), \qquad \Psi_\lambda : \tilde{U}^-_\zeta \to M^*_{-,\zeta}(\lambda)$$

the isomorphisms of $\mathbb{C}$-modules given by

$$\langle \Phi_\lambda(x), \overline{v} \rangle = \tau^L_\zeta(x, v) \qquad (x \in U^+_\zeta, v \in \tilde{U}^{L,-}_\zeta),$$

$$\langle \Psi_\lambda(y), \overline{Su} \rangle = {}^L\tau_\zeta(u, y) \qquad (y \in \tilde{U}^-_\zeta, u \in U^{L,+}_\zeta).$$

By Lemma 3.5 we have the following.

Lemma **3.6.** (i) *The U^L_ζ-module structure of $M^*_{+,\zeta}(\lambda)$ is given by*

$$h \cdot \Phi_\lambda(x) = \chi_{\lambda+\gamma}(h)\Phi_\lambda(x) \quad (x \in U^+_{\zeta,\gamma}, h \in U^{L,0}_\zeta), \tag{3.27}$$

$$v \cdot \Phi_\lambda(x) = \sum_{(x)} \tau^L_\zeta(x_{(0)}, Sv)\Phi_\lambda(x_{(1)}) \quad (x \in U^+_\zeta, v \in U^{L,-}_\zeta), \tag{3.28}$$

$$u \cdot \Phi_\lambda(x) = \Phi_\lambda(k_{-\lambda}(\mathrm{ad}(u)(k_\lambda x k_\lambda))k_{-\lambda}) \quad (x \in U^+_\zeta,\ u \in U^{L,+}_\zeta). \tag{3.29}$$

(ii) *The U^L_ζ-module structure of $M^*_{-,\zeta}(\lambda)$ is given by*

$$h \cdot \Psi_\lambda(y) = \chi_{\lambda-\gamma}(h)\Psi_\lambda(y) \quad (y \in \tilde{U}^-_{\zeta,-\gamma}, h \in U^{L,0}_\zeta), \tag{3.30}$$

$$u \cdot \Psi_\lambda(y) = \sum_{(y)} {}^L\tau_\zeta(u, y_{(0)})\Psi_\lambda(y_{(1)}) \quad (y \in \tilde{U}^-_\zeta, u \in U^{L,+}_\zeta), \tag{3.31}$$

$$v \cdot \Psi_\lambda(y) = \Psi_\lambda(k_\lambda(\mathrm{ad}(v)(k_{-\lambda} y k_{-\lambda}))k_\lambda) \quad (y \in \tilde{U}^-_\zeta,\ v \in U^{L,-}_\zeta). \tag{3.32}$$

For $\lambda \in \Lambda^+$ we set

$$U^+_\zeta(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} U^+_{\mathbb{A}}(\lambda), \quad \tilde{U}^-_\zeta(\lambda) = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}^-_{\mathbb{A}}(\lambda).$$

Then $U^+_\zeta(\lambda)$ and $\tilde{U}^-_\zeta(\lambda)$ are the $\mathbb{C}$-submodules of U^+_ζ and $\tilde{U}^-_\zeta$ respectively satisfying $\Phi_{-\lambda}(U^+_\zeta(\lambda)) = L^*_{+,\zeta}(-\lambda)$ and $\Psi_\lambda(\tilde{U}^-_\zeta(\lambda)) = L^*_{-,\zeta}(\lambda)$. We have linear isomorphisms

$$\Phi_{-\lambda} : U^+_\zeta(\lambda) \to L^*_{+,\zeta}(-\lambda), \quad \Psi_\lambda : \tilde{U}^-_\zeta(\lambda) \to L^*_{-,\zeta}(\lambda) \quad (\lambda \in \Lambda^+). \tag{3.33}$$

By (3.21), (3.22), (3.23) we have

$$U_\zeta^+(\lambda) \subset U_\zeta^+(\lambda+\mu), \quad \tilde{U}_\zeta^-(\lambda) \subset \tilde{U}_\zeta^-(\lambda+\mu) \qquad (\lambda, \mu \in \Lambda^+), \tag{3.34}$$

$$U_\zeta^+ = \sum_{\lambda\in\Lambda^+} U_\zeta^+(\lambda), \quad \tilde{U}_\zeta^- = \sum_{\lambda\in\Lambda^+} \tilde{U}_\zeta^-(\lambda), \tag{3.35}$$

$$\tilde{U}_\zeta^-(\lambda)k_{-2\lambda} \subset U_{\zeta,f}, \quad k_{-2\lambda}U_\zeta^+(\lambda) \subset U_{\mathbb{A},f} \qquad (\lambda \in \Lambda^+). \tag{3.36}$$

By (3.35), (3.36) we see easily the following.

LEMMA **3.7.** *For any $u \in U_\zeta$ there exists some $\lambda \in \Lambda^+$ such that $uk_{-2\lambda} \in U_{\zeta,f}$.*

4. Induction functor

4.1. **Functors.** We set

$$C_\zeta^{\leqq 0} = C_\zeta/I, \qquad I = \{\varphi \in C_\zeta \mid \langle \varphi, U_\zeta^{L,\leqq 0}\rangle = \{0\}\}.$$

Then $C_\zeta^{\leqq 0}$ is a Hopf algebra and we have a Hopf paring

$$\langle\ ,\ \rangle : C_\zeta^{\leqq 0} \times U_\zeta^{L,\leqq 0} \to \mathbb{C}.$$

We have a canonical Hopf algebra homomorphism

$$\mathrm{res} : C_\zeta \to C_\zeta^{\leqq 0}.$$

Following Backelin-Kremnizer [2] we define abelian categories $\mathcal{M}_\zeta$ and $\mathcal{M}_\zeta^{eq}$ as follows.

An object of $\mathcal{M}_\zeta$ is a triplet (M, α, β) with

(1) M is a vector space over $\mathbb{C}$,
(2) $\alpha : C_\zeta \otimes M \to M$ is a left C_ζ-module structure of M,
(3) $\beta : M \to C_\zeta^{\leqq 0} \otimes M$ is a left $C_\zeta^{\leqq 0}$-comodule structure of M

such that β is a morphism of C_ζ-modules (or equivalently, α is a morphism of $C_\zeta^{\leqq 0}$-comodules). A morphism from (M, α, β) to (M', α', β') is a linear map $\varphi : M \to M'$ which is a morphism of C_ζ-modules as well as that of $C_\zeta^{\leqq 0}$-comodules.

An object of $\mathcal{M}_\zeta^{eq}$ is a quadruple $(M, \alpha, \beta, \gamma)$ with

(1) M is a vector space over $\mathbb{C}$,
(2) $\alpha : C_\zeta \otimes M \to M$ is a left C_ζ-module structure of M,
(3) $\beta : M \to C_\zeta^{\leqq 0} \otimes M$ is a left $C_\zeta^{\leqq 0}$-comodule structure of M,
(4) $\gamma : M \to M \otimes C_\zeta$ is a right C_ζ-comodule structure of M

subject to the conditions that $(M, \alpha, \beta) \in \mathcal{M}_\zeta$, β and γ commutes with each other, and γ is a homomorphism of left C_ζ-modules. A morphism from $(M, \alpha, \beta, \gamma)$ to $(M', \alpha', \beta', \gamma')$ is a linear map $\varphi : M \to M'$ which is compatible with the left C_ζ-module structure, the left $C_\zeta^{\leqq 0}$-comodule structure and the right C_ζ-comodule structure.

For a coalgebra $\mathcal{C}$ we denote by $\mathrm{Comod}(\mathcal{C})$ (resp. $\mathrm{Comod}^r(\mathcal{C})$) the category of left $\mathcal{C}$-comodules (resp. right $\mathcal{C}$-comodules). We define functors

$$\Xi : \mathcal{M}_\zeta^{eq} \to \mathrm{Comod}(C_\zeta^{\leqq 0}),$$
$$\Upsilon : \mathrm{Comod}(C_\zeta^{\leqq 0}) \to \mathcal{M}_\zeta^{eq}$$

by

$$\Xi(M) = \{M \in M \mid \gamma(m) = m \otimes 1\},$$
$$\Upsilon(L) = C_\zeta \otimes L.$$

By Backelin-Kremnizer [2] we have

Proposition **4.1.** *The functor* $\Xi : \mathcal{M}_\zeta^{eq} \to \mathrm{Comod}(C_\zeta^{\leqq 0})$ *gives an equivalence of categories, and its quasi-inverse is given by* Υ.

Remark **4.2.** For $M \in \mathcal{M}_\zeta^{eq}$ we have an isomorphism

$$\Xi(M) \cong \mathbb{C} \otimes_{C_\zeta} M$$

of vector spaces by Proposition 4.1. Here $C_\zeta \to \mathbb{C}$ is given by ε.

For $\lambda \in \Lambda$ we define $\chi_\lambda^{\leqq 0} \in C_\zeta^{\leqq 0} \subset \mathrm{Hom}_{\mathbb{C}}(U_\zeta^{L,\leqq 0}, \mathbb{C})$ by

$$\chi_\lambda^{\leqq 0}(hu) = \chi_\lambda(h)\varepsilon(u) \qquad (h \in U_\zeta^{L,0}, u \in U_\zeta^{L,-}).$$

We define left exact functors

$$\omega_{\mathcal{M}*} : \mathcal{M}_\zeta \to \mathrm{Mod}_\Lambda(A_\zeta), \tag{4.1}$$
$$\Gamma_{\mathcal{M}} : \mathcal{M}_\zeta \to \mathrm{Mod}(\mathbb{C}) \tag{4.2}$$

by

$$\omega_{\mathcal{M}*}(M) = \bigoplus_{\lambda \in \Lambda} (\omega_{\mathcal{M}*}(M))(\lambda) \subset M,$$
$$(\omega_{\mathcal{M}*}(M))(\lambda) = \{m \in M \mid \beta(m) = \chi_\lambda^{\leqq 0} \otimes m\},$$
$$\Gamma_{\mathcal{M}}(M) = (\omega_{\mathcal{M}*}(M))(0).$$

We denote by $\mathrm{Mod}_\Lambda^{eq}(A_\zeta)$ the category consisting of $N \in \mathrm{Mod}_\Lambda(A_\zeta)$ equipped with a right C_ζ-comodule structure $\gamma : N \to N \otimes C_\zeta$ such that $\gamma(N(\lambda)) \subset N(\lambda) \otimes C_\zeta$ for any $\lambda \in \Lambda$ and $\gamma(\varphi n) = \Delta(\varphi)\gamma(n)$ for any $\varphi \in A_\zeta$ and $n \in N$ (note that $\Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta$). By definition (4.1), (4.2) induce left exact functors

$$\omega_{\mathcal{M}*}^{eq} : \mathcal{M}_\zeta^{eq} \to \mathrm{Mod}_\Lambda^{eq}(A_\zeta), \tag{4.3}$$
$$\Gamma_{\mathcal{M}}^{eq} : \mathcal{M}_\zeta^{eq} \to \mathrm{Comod}^r(C_\zeta). \tag{4.4}$$

We also define a left exact functor

$$\mathrm{Ind} : \mathrm{Comod}(C_\zeta^{\leqq 0}) \to \mathrm{Comod}^r(C_\zeta). \tag{4.5}$$

by $\mathrm{Ind} = \Gamma_{\mathcal{M}}^{eq} \circ \Upsilon$.

The abelian categories $\mathcal{M}_\zeta$, $\mathcal{M}_\zeta^{eq}$, $\mathrm{Comod}^r(C_\zeta)$ have enough injectives, and the forgetful functor $\mathcal{M}_\zeta^{eq} \to \mathcal{M}_\zeta$ sends injective objects to

$\Gamma_{\mathcal{M}}$-accyclic objects (see Backelin-Kremnizer [2, 3.4]). Hence we have the following.

Lemma **4.3.** *We have*

$$\begin{aligned}\mathrm{For}\circ R^i\Gamma^{eq}_{\mathcal{M}} =& R^i\Gamma_{\mathcal{M}}\circ\mathrm{For}:\mathcal{M}^{eq}_{\zeta}\to\mathrm{Mod}(\mathbb{C}),\\ R^i\,\mathrm{Ind}\circ\Xi =& R^i\Gamma^{eq}_{\mathcal{M}}:\mathcal{M}^{eq}_{\zeta}\to\mathrm{Comod}^r(C_\zeta).\end{aligned}$$

for any i, where $\mathrm{For}:\mathrm{Comod}^r(C_\zeta)\to\mathrm{Mod}(\mathbb{C})$ *and* $\mathrm{For}:\mathcal{M}^{eq}_{\zeta}\to\mathcal{M}_\zeta$ *are forgetful functors.*

We define an exact functor

$$\mathrm{res}:\mathrm{Comod}^r(C_\zeta)\to\mathrm{Comod}(C^{\leqq 0}_{\zeta}) \tag{4.6}$$

as follows. For $V\in\mathrm{Comod}^r(C_\zeta)$ with right C_ζ-comodule structure $\beta:V\to V\otimes C_\zeta$ we have $\mathrm{res}(V)=V$ as a $\mathbb{C}$-module and the left $C^{\leqq 0}_{\zeta}$-comodule structure $\mathrm{res}(V)\to C^{\leqq 0}_{\zeta}\otimes\mathrm{res}(V)$ of $\mathrm{res}(V)$ is given by

$$\beta(v)=\sum_k v_k\otimes\varphi_k\quad\Longrightarrow\quad\gamma(v)=\sum_k\mathrm{res}(S^{-1}\varphi_k)\otimes v_k.$$

The following fact is standard.

Lemma **4.4.** *For $V\in\mathrm{Comod}^r(C_\zeta)$, $M\in\mathrm{Comod}(C^{\leqq 0}_{\zeta})$ we have an isomorphism*

$$F:\mathrm{Ind}(M)\otimes V\to\mathrm{Ind}(\mathrm{res}(V)\otimes M)$$

of right C_ζ-comodules given by

$$F((\sum_i\varphi_i\otimes m_i)\otimes v)=\sum_{i,(v)}\varphi_i v_{(1)}\otimes v_{(0)}\otimes m_i,$$

where we write the right C_ζ-comodule structure of V by

$$V\ni v\mapsto\sum_{(v)}v_{(0)}\otimes v_{(1)}\in V\otimes C_\zeta.$$

For $\lambda\in\Lambda$ we denote by $\mathbb{C}^{\leqq 0}_{\lambda}=\mathbb{C}1^{\leqq 0}_{\lambda}$ the object of $\mathrm{Comod}(C^{\leqq 0}_{\zeta})$ corresponding to the one-dimensional right $U^{L,\leqq 0}_{\zeta}$-module given by $1^{\leqq 0}_{\lambda}u=\chi^{\leqq 0}_{\lambda}(u)1^{\leqq 0}_{\lambda}$ for $u\in U^{L,\leqq 0}_{\zeta}$. By definition we have an isomorphism

$$\mathrm{Ind}(\mathbb{C}^{\leqq 0}_{-\lambda})\cong A_\zeta(\lambda)\qquad(\lambda\in\Lambda^+)$$

of right C_ζ-comodules.

Let $N\in\mathrm{Mod}_\Lambda(A_\zeta)$. Then $C_\zeta\otimes_{A_\zeta}N$ turns out to be an object of $\mathcal{M}_\zeta$ by

$$\alpha(f\otimes(f'\otimes n)=ff'\otimes n\qquad(f,f'\in C_\zeta,\,n\in N),$$

$$\beta(f\otimes n)=\sum_{(f)}\mathrm{res}(f_{(0)})\chi_\lambda\otimes(f_{(1)}\otimes n)\qquad(f\in C_\zeta,\ n\in N(\lambda)).$$

Hence we have a functor $\mathrm{Mod}_\Lambda(A_\zeta)\to\mathcal{M}_\zeta$ sending N to $C_\zeta\otimes_{A_\zeta}N$.

LEMMA **4.5.** *The functor* $\mathrm{Mod}_\Lambda(A_\zeta) \to \mathcal{M}_\zeta$ *as above induces a functor*

$$\Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}_\zeta.$$

PROOF. It is sufficient to show $C_\zeta \otimes_{A_\zeta} A_\zeta / A_\zeta(\lambda + \Lambda^+) = \{0\}$ for any $\lambda \in \Lambda$. Hence we have only to show $C_\zeta A_\zeta(\lambda) = C_\zeta$ for any $\lambda \in \Lambda^+$. Take $\varphi \in A_\zeta(\lambda)$ such that $\varepsilon(\varphi) = 1$. We have $\Delta(A_\zeta(\lambda)) \subset A_\zeta(\lambda) \otimes C_\zeta$ and hence we can write $\Delta(\varphi) = \sum_i \varphi_i \otimes \varphi'_i$ with $\varphi_i \in A_\zeta(\lambda)$, $\varphi'_i \in C_\zeta$. Then we have $C_\zeta A_\zeta(\lambda) \ni \sum_i (S^{-1}\varphi'_i)\varphi_i = 1$. □

We set

$$\Psi = \omega^* \circ \omega_{\mathcal{M}*} : \mathcal{M}_\zeta \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}).$$

Backelin-Kremnizer [2] obtained the following result using a result of Artin-Zhang [1].

PROPOSITION **4.6.** *The functor* $\Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}_\zeta$ *gives an equivalence of categories, and its quasi-inverse is given by* Ψ*. Moreover, we have an identification*

$$\omega_{\mathcal{M}*} \circ \Phi = \omega_* : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}_\Lambda(A_\zeta).$$

of functors.

Hence we have the following.

LEMMA **4.7.** *We have*

$$R^i\Gamma = R^i\Gamma_{\mathcal{M}} \circ \Phi : \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C})$$

for any i*.*

We set

$$\mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) = \mathrm{Mod}^{eq}_\Lambda(A_\zeta) / \mathrm{Mod}^{eq}_\Lambda(A_\zeta) \cap \mathrm{Tor}_{\Lambda^+}(A_\zeta).$$

Let $N \in \mathrm{Mod}^{eq}_\Lambda(A_\zeta)$. We denote the right C_ζ-comodule structure of N by $\gamma' : N \to N \otimes C_\zeta$. Then we have a right C_ζ-comodule structure $\gamma : C_\zeta \otimes_{A_\zeta} N \to (C_\zeta \otimes_{A_\zeta} N) \otimes C_\zeta$ of $C_\zeta \otimes_{A_\zeta} N$ given by

$$\gamma'(n) = \sum_k n_k \otimes \varphi_k \quad \Longrightarrow \quad \gamma(f \otimes n) = \sum_{k,(f)} (f_{(0)} \otimes n_k) \otimes f_{(1)}\varphi_k$$

This gives a functor $\mathrm{Mod}^{eq}_\Lambda(A_\zeta) \to \mathcal{M}^{eq}_\zeta$. Hence by Lemma 4.5 we have a functor

$$\Phi^{eq} : \mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}^{eq}_\zeta \tag{4.7}$$

induced by Φ. Let $M \in \mathcal{M}^{eq}_\zeta$. The right C_ζ-comodule structure of M restricts to that of $\omega_{\mathcal{M}*}M$ so that $\omega_{\mathcal{M}*}M \in \mathrm{Mod}^{eq}_\Lambda(A_\zeta)$. Hence we have a functor

$$\Psi^{eq} : \mathcal{M}^{eq}_\zeta \to \mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) \tag{4.8}$$

induced by Ψ. By Proposition 4.6 we have the following.

PROPOSITION **4.8.** *The functor* $\Phi^{eq} : \mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathcal{M}^{eq}_\zeta$ *gives an equivalence of categories, and its quasi-inverse is given by* Ψ^{eq}*.*

By Proposition 4.8 we see that (4.1), (4.2) induce

$$\omega_*^{eq} = \omega_{\mathcal{M}*}^{eq} \circ \Phi^{eq} : \mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}_\Lambda^{eq}(A_\zeta), \tag{4.9}$$

$$\Gamma^{eq} = \Gamma_{\mathcal{M}}^{eq} \circ \Phi^{eq} : \mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Comod}^r(C_\zeta). \tag{4.10}$$

By Lemma 4.3 we have the following.

Lemma **4.9.** *We have*

$$\mathrm{For} \circ R^i\Gamma^{eq} = R^i\Gamma \circ \mathrm{For} : \mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathbb{C})$$

for any i*, where* $\mathrm{For} : \mathrm{Comod}^r(C_\zeta) \to \mathrm{Mod}(\mathbb{C})$ *and* $\mathrm{For} : \mathrm{Mod}^{eq}(\mathcal{O}_{\mathcal{B}_\zeta}) \to \mathrm{Mod}(\mathcal{O}_{\mathcal{B}_\zeta})$ *are forgetful functors.*

5. Reformulation of Conjecture 2.14

5.1. **adjoint action of U_ζ^L on D'_ζ.** Define a left $U_\mathbb{F}$-module structure of $E_\mathbb{F}$ by

$$\mathrm{ad}(u)(P) = \sum_{(u)} u_{(0)} P(S u_{(1)}) \qquad (u \in U_\mathbb{F},\ P \in E_\mathbb{F}).$$

Then we have

$$\mathrm{ad}(u)(P_1P_2) = \sum_{(u)} \mathrm{ad}(u_{(0)})(P_1)\,\mathrm{ad}(u_{(1)})(P_2) \qquad (P_1, P_2 \in E_\mathbb{F}),$$

$$\begin{aligned} \mathrm{ad}(u)(\varphi) &= u \cdot \varphi \quad (\varphi \in A_\mathbb{F} \subset E_\mathbb{F}), \\ \mathrm{ad}(u)(v) &= \sum_{(u)} u_{(0)} v(S u_{(1)}) \quad (v \in U_\mathbb{F} \subset E_\mathbb{F}), \\ \mathrm{ad}(u)(e(\lambda)) &= \varepsilon(u) e(\lambda) \quad (\lambda \in \Lambda, e(\lambda) \in \mathbb{F}[\Lambda] \subset E_\mathbb{F}) \end{aligned}$$

for $u \in U_\mathbb{F}$. We see from [20, Lemma 4.2] that this induces a left $U_\mathbb{F}$-module structure of $D'_\mathbb{F}$. Moreover, the $U_\mathbb{F}$-module structures of $E_\mathbb{F}$ and $D'_\mathbb{F}$ induce $U_\mathbb{A}^L$-module structures of $E_\mathbb{A}$, $D'_\mathbb{A}$, $E_{\mathbb{A},\diamondsuit}$, $D'_{\mathbb{A},\diamondsuit}$, $E_{\mathbb{A},f}$, $D'_{\mathbb{A},f}$ by Lemma 1.2 and Lemma 2.12. Hence by specialization we obtain U_ζ^L-module structures of E_ζ, D'_ζ, $E_{\zeta,\diamondsuit}$, $D'_{\zeta,\diamondsuit}$, $E_{\zeta,f}$, $D'_{\zeta,f}$ also denoted by ad.

5.2. We will regard $E_{\zeta,f}, D'_{\zeta,f} \in \mathrm{Mod}_\Lambda(A_\zeta)$ as objects of $\mathrm{Mod}_\Lambda^{eq}(A_\zeta)$ by the right C_ζ-comodule structures induced from the left U_ζ^L-module structures

$$(u, P) \mapsto \mathrm{ad}(u)(P) \qquad (u \in U_\zeta^L, P \in E_{\zeta,f} \text{ or } D'_{\zeta,f}).$$

Then for

$$(\Xi \circ \Phi^{eq})\,(\omega^* D'_{\zeta,f}) \in \mathrm{Comod}(C_\zeta^{\leqq 0})$$

we have

$$R^i\Gamma(\omega^* D'_{\zeta,f}) = R^i\,\mathrm{Ind}((\Xi \circ \Phi^{eq})\,(\omega^* D'_{\zeta,f}))$$

by Lemma 4.3, Lemma 4.9, and (4.10).

Define a right $(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])$-module V by

$$V=(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])/\mathcal{I}$$

where

$$\mathcal{I}=(\tilde{U}_\zeta^-\cap\operatorname{Ker}(\varepsilon))U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda]+\sum_{\lambda\in\Lambda}(k_{2\lambda}-e(2\lambda))U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda].$$

By the triangular decomposition $\tilde{U}_\zeta^-\otimes U_{\zeta,\diamondsuit}^0\otimes U_\zeta^+\cong U_{\zeta,\diamondsuit}$ we have

$$V\cong U_\zeta^+\otimes\mathbb{C}[\Lambda]$$

as a vector space. Define a right action of $U_\zeta^{L,\leqq 0}$ on $U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda]$ by

$$(u\otimes e(\lambda))\star v=\operatorname{ad}(Sv)(u)\otimes e(\lambda)\qquad(u\in U_{\zeta,\diamond},\lambda\in\Lambda,v\in U_\zeta^{L\leqq 0}).$$

It induces a right action of $U_\zeta^{L,\leqq 0}$ on V. Moreover, we see easily that this right $U_\zeta^{L,\leqq 0}$-module structure gives a left $C_\zeta^{\leqq 0}$-comodule structure of V.

Proposition **5.1.** *We have*

$$(\Xi\circ\Phi^{eq})\,(\omega^*D'_{\zeta,f})\cong V$$

as a left $C_\zeta^{\leqq 0}$-comodule.

The proof is given in the next subsection.

It follows from Proposition 5.1 that Conjecture 2.14 is equivalent to the following conjecture.

Conjecture **5.2.** *Assume $\ell>h_G$. We have*

$$\operatorname{Ind}(V)\cong U_{\zeta,f}\otimes_{Z_{Har}(U_\zeta)}\mathbb{C}[\Lambda],$$

and

$$R^i\operatorname{Ind}(V)=0$$

for $i\neq 0$.

Remark **5.3.** We can show that

$$U_{\zeta,f}\cong(C_\zeta)_{\mathrm{ad}},\qquad V\cong{}_{\mathrm{ad}}(C_\zeta^{\leqq 0})\otimes_{\mathbb{C}[2\Lambda]}\mathbb{C}[\Lambda],$$

where $(C_\zeta)_{\mathrm{ad}}$ (resp. ${}_{\mathrm{ad}}(C_\zeta^{\leqq 0})$) is given by the right (resp. left) adjoint coaction of C_ζ (resp. $C_\zeta^{\leqq 0}$) on itself. Hence Conjecture 5.2 is equivalent to

$$R\operatorname{Ind}({}_{\mathrm{ad}}(C_\zeta^{\leqq 0}))\cong(C_\zeta)_{\mathrm{ad}}\otimes_{\mathbb{C}[2\Lambda]^W}\mathbb{C}[2\Lambda].$$

The corresponding statement for $q=1$ is

$$R\operatorname{Ind}({}_{\mathrm{ad}}\mathbb{C}[B^-])\cong\mathbb{C}[G]_{\mathrm{ad}}\otimes_{\mathbb{C}[H/W]}\mathbb{C}[H].$$

We can prove this by a geometric method.

5.3. We will give a proof of Proposition 5.1 in the rest of this paper. By Remark 4.2. we have

$$(\Xi \circ \Phi^{eq})\,(\omega^* D'_{\zeta,f}) \cong \mathbb{C} \otimes_{A_\zeta} D'_{\zeta,f}$$

as a vector space, where $A_\zeta \to \mathbb{C}$ is given by ε. Note that

$$\mathbb{C} \otimes_{A_\zeta} E_{\zeta,\diamondsuit} \cong U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda].$$

We first show the following.

Lemma **5.4.** *We have*

$$\mathbb{C} \otimes_{A_\zeta} D'_{\zeta,\diamondsuit} \cong V.$$

Proof. By (2.10) we obtain

$$\mathbb{C} \otimes_{A_\zeta} D'_{\zeta,\diamondsuit} \cong (U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]) / \sum_{\varphi \in A_\zeta} (1 \otimes \Omega'(\varphi))(U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]),$$

where $1 \otimes \Omega'(\varphi)$ is the image of $\Omega'(\varphi)$ in $\mathbb{C} \otimes_{A_\zeta} E_{\zeta,\diamondsuit} = U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$. Note that $\varepsilon(A_\zeta(\lambda)_\xi) = \{0\}$ for $\lambda \in \Lambda^+$, $\xi \in \Lambda$ with $\lambda \neq \xi$, and $\varepsilon(A_\zeta(\lambda)_\lambda) = \mathbb{C}$ for $\lambda \in \Lambda^+$. Hence for $\varphi \in A_\zeta(\lambda)_\xi$ with $\lambda \in \Lambda^+$, $\xi \in \Lambda$ we have

$$1 \otimes \Omega'_1(\varphi) = \begin{cases} 0 & (\lambda \neq \xi) \\ \varepsilon(\varphi) & (\lambda = \xi). \end{cases}$$

Let us also compute $1 \otimes \Omega'_2(\varphi)$. Let

$$\tilde{\Psi}_\lambda : \tilde{U}_\zeta^-(\lambda) \to A_\zeta(\lambda)$$

be the composite of the linear isomorphism $\Psi_\lambda : \tilde{U}_\zeta^-(\lambda) \to L^*_{-,\zeta}(\lambda)$ (see (3.33)) and an isomorphism $f : L^*_{-,\zeta}(\lambda) \to A_\zeta(\lambda)$ of U_ζ^L-modules. We have $\tilde{\Psi}_\lambda(\tilde{U}_\zeta^-(\lambda)_{-(\lambda-\xi)}) = A_\zeta(\lambda)_\xi$ for any $\xi \in \Lambda$. Hence we may assume $\varepsilon = \varepsilon \circ \tilde{\Psi}_\lambda$ on $\tilde{U}_\zeta^-(\lambda)$. Let $\varphi \in A_\zeta(\lambda)_\xi$ and take $v \in \tilde{U}_\zeta^-(\lambda)_{-(\lambda-\xi)}$ satisfying $\tilde{\Psi}_\lambda(v) = \varphi$. Then we have

$$\begin{aligned}
&\sum_p (Sx_p^L) \cdot \varphi \otimes y_p k_{\beta_p} \\
=&\sum_p f((Sx_p^L) \cdot \Psi_\lambda(v)) \otimes y_p k_{\beta_p} \\
=&\sum_p \zeta^{-(\beta_p,\xi)} f((Sx_p^L) k_{\beta_p} \cdot \Psi_\lambda(v)) \otimes y_p k_{\beta_p} \\
=&\sum_{p,(v)} \zeta^{-(\beta_p,\xi)} f({}^L\tau_\zeta((Sx_p^L)k_{\beta_p}, v_{(0)}) \Psi_\lambda(v_{(1)})) \otimes y_p k_{\beta_p} \\
=&\sum_{p,(v)} \zeta^{-(\beta_p,\xi)}\, {}^L\tau_\zeta((Sx_p^L)k_{\beta_p}, v_{(0)}) \tilde{\Psi}_\lambda(v_{(1)}) \otimes y_p k_{\beta_p},
\end{aligned}$$

and hence

$$\begin{aligned}
&1\otimes\Omega_2'(\varphi)\\
=&\sum_p \varepsilon((Sx_p^L)\cdot\varphi)y_pk_{\beta_p}k_{2\xi}e(-2\lambda)\\
=&\sum_{p,(v)} \zeta^{-(\beta_p,\xi)L}\tau_\zeta((Sx_p^L)k_{\beta_p},v_{(0)})\varepsilon(v_{(1)})y_pk_{\beta_p}k_{2\xi}e(-2\lambda)\\
=&\sum_p \zeta^{-(\beta_p,\xi)L}\tau_\zeta((Sx_p^L)k_{\beta_p},v)y_pk_{\beta_p}k_{2\xi}e(-2\lambda)\\
=&\sum_p \zeta^{-(\beta_p,\xi)L}\tau_\zeta(k_{-\beta_p}x_p^L,S^{-1}v)y_pk_{\beta_p}k_{2\xi}e(-2\lambda)\\
=&\sum_p \zeta^{-(\beta_p,\xi)-(\beta_p,\beta_p)L}\tau_\zeta(x_p^L,S^{-1}v)y_pk_{\beta_p}k_{2\xi}e(-2\lambda)\\
=&\sum_p \zeta^{-(\lambda-\xi,\lambda)L}\tau_\zeta(x_p^L,S^{-1}v)y_pk_{\lambda-\xi}k_{2\xi}e(-2\lambda)\\
=&\zeta^{-(\lambda-\xi,\lambda)}(S^{-1}v)k_{\lambda-\xi}k_{2\xi}e(-2\lambda)
\end{aligned}$$

(note $(S^{-1}v)k_{\lambda-\xi}\in\tilde{U}_\zeta^-(\lambda)_{-(\lambda-\xi)}$). It follows that

$$1\otimes\Omega'(\varphi)=\begin{cases}-\zeta^{-(\lambda-\xi,\lambda)}(S^{-1}v)k_{\lambda-\xi}k_{2\xi}e(-2\lambda) & (\lambda\neq\xi)\\ \varepsilon(\varphi)(1-k_{2\lambda}e(-2\lambda)) & (\lambda=\xi).\end{cases}$$

Hence we have

$$\begin{aligned}
&\sum_{\substack{\lambda\in\Lambda^+,\\ \gamma\in Q^+}}\ \sum_{\varphi\in A_\zeta(\lambda)_{\lambda-\gamma}}(1\otimes\Omega'(\varphi))(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])\\
=&\sum_{\substack{\lambda\in\Lambda^+,\\ \gamma\in Q^+\setminus\{0\}}}\tilde{U}_\zeta^-(\lambda)_{-\gamma}(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])+\sum_{\lambda\in\Lambda^+}(1-k_{2\lambda}e(-2\lambda))(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])\\
=&(\tilde{U}_\zeta^-\cap\mathrm{Ker}(\varepsilon))(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])+\sum_{\lambda\in\Lambda}(k_{2\lambda}-e(2\lambda))(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])
\end{aligned}$$

by (3.35). □

Lemma **5.5.** *We have*

$$\mathbb{C}\otimes_{A_\zeta}D'_{\zeta,f}\cong V.$$

Proof. We need to show that the canonical homomorphism $\mathbb{C}\otimes_{A_\zeta}D'_{\zeta,f}\to\mathbb{C}\otimes_{A_\zeta}D'_{\zeta,\diamondsuit}$ is bijective. The surjectivity is a consequence of (3.35), (3.36). Let us give a proof of the injectivity. Set

$$\mathcal{K}=A_\zeta U_{\zeta,f}\mathbb{C}[\Lambda]\cap\sum_{\varphi\in A_\zeta}A_\zeta\Omega'(\varphi)U_{\zeta,\diamondsuit}\mathbb{C}[\Lambda]\subset A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda].$$

Then it is sufficient to show that the natural map

$$\mathbb{C}\otimes_{A_\zeta}((A_\zeta\otimes U_{\zeta,f}\otimes\mathbb{C}[\Lambda])/\mathcal{K})\to(U_{\zeta,\diamondsuit}\otimes\mathbb{C}[\Lambda])/\mathcal{I}$$

is injective. Let $F : A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \to U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$ be the natural map. Then it is sufficient to show

$$\mathcal{I} \cap (U_{\zeta,f} \otimes \mathbb{C}[\Lambda]) \subset F(\mathcal{K}). \tag{5.1}$$

Indeed, assume that (5.1) holds. Denote by

$$p : A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \to \mathbb{C} \otimes_{A_\zeta} ((A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda])/\mathcal{K}),$$
$$\pi : U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda] \to (U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda])/\mathcal{I}$$

the natural maps. We have to show $\mathrm{Ker}(\pi \circ F) \subset \mathrm{Ker}(p)$. Take $x \in \mathrm{Ker}(\pi \circ F)$. Then $F(x) \in \mathcal{I} \cap (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$. Hence by (5.1) there exists some $v \in \mathcal{K}$ such that $F(x) = F(v)$. Then $p(x) = p(x - v) + p(v) = p(x - v)$. Hence we may assume that $F(x) = 0$ from the beginning. Note that p factors through

$$p' : A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \to \mathbb{C} \otimes_{A_\zeta} (A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda])(= U_{\zeta,f} \otimes \mathbb{C}[\Lambda]).$$

By $F(x) = 0$ we have $p'(x) = 0$ and hence $p(x) = 0$ as desired.

It remains to show (5.1). Let $\lambda \in \Lambda^+$ and $\varphi \in A_\zeta(\lambda)_\lambda$. Then we have

$$\Omega'_1(\varphi) = \sum_p (y_p^L \cdot \varphi) x_p \in A_\zeta U_\zeta^+, \qquad \Omega'_2(\varphi) = \varphi k_{2\lambda} e(-2\lambda).$$

Let us show

$$\Omega'_1(\varphi) = \sum_p (y_p^L \cdot \varphi) x_p \in A_\zeta U_\zeta^+(\lambda). \tag{5.2}$$

This is equivalent to

$$\sum_p (y_p^L \cdot \varphi) \otimes \Phi_{-\lambda}(x_p) \in A_\zeta \otimes L^*_{+,\zeta}(-\lambda).$$

This follows from

$$\sum_p \left\langle \Phi_{-\lambda}(x_p), \overline{u f_i^{((\lambda,\alpha_i^\vee)+1)}} \right\rangle y_p^L \cdot \varphi = \sum_p \tau_\zeta^L \left(x_p, u f_i^{((\lambda,\alpha_i^\vee)+1)} \right) y_p^L \cdot \varphi$$
$$= (u f_i^{((\lambda,\alpha_i^\vee)+1)}) \cdot \varphi = 0$$

for $u \in U_\zeta^{L,-}, i \in I$. (5.2) is verified. Hence we have

$$\Omega'(\varphi) k_{-2\lambda} \in \mathcal{K}.$$

It follows that

$$F(\mathcal{K}) \supset (k_{-2\lambda} - e(-2\lambda)) U_{\zeta,f} \mathbb{C}[\Lambda] \qquad (\lambda \in \Lambda^+). \tag{5.3}$$

Now let $u \in \mathcal{I} \cap (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$. If we can show that $k_{-2\mu} u \in F(\mathcal{K})$ for some $\mu \in \Lambda^+$, then we obtain

$$u = e(2\mu)(e(-2\mu) - k_{-2\mu}) u + e(2\mu) k_{-2\mu} u \in F(\mathcal{K})$$

by (5.3). Hence it is sufficient to show that for any $u \in \mathcal{I}$ there exists some $\mu \in \Lambda^+$ such that $k_{-2\mu} u \in F(\mathcal{K})$. We may assume that there exists $\nu \in Q$ such that $k_{-2\mu} u = \zeta^{(\mu,\nu)} u k_{-2\mu}$ for any $\mu \in \Lambda$. Therefore, we have only to show that for any $u \in \mathcal{I}$ there exists some $\mu \in \Lambda^+$

such that $uk_{-2\mu} \in F(\mathcal{K})$. By Lemma 5.4 we can take $\varphi_i \in A_\zeta$, $x_i \in U_{\zeta,\diamondsuit} \otimes \mathbb{C}[\Lambda]$ $(i = 1, \dots, N)$ such that

$$u = 1 \otimes \sum_{i=1}^{N} \Omega'(\varphi_i)x_i.$$

By Lemma 3.7 we can take $\mu \in \Lambda^+$ such that $\Omega'(\varphi_i)x_i k_{-2\mu} \in A_\zeta \otimes U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ for any i. Then we have

$$uk_{-2\mu} = \sum_{i=1}^{N} F(\Omega'(\varphi_i)x_i k_{-2\mu}) \in F(\mathcal{K}).$$

□

By Lemma 5.5 we obtain an isomorphism

$$(\Xi \circ \Phi^{eq})\,(\omega^* D'_{\zeta,f}) \cong V$$

of vector spaces. We need to show that it is in fact an isomorphism of left $C_\zeta^{\leqq 0}$-comodules. This is a consequence of the corresponding fact for $E_{\zeta,f}$. Note that we have

$$\mathbb{C} \otimes_{A_\zeta} E_{\zeta,f} \cong U_{\zeta,f} \otimes \mathbb{C}[\Lambda],$$

and hence we have an isomorphism

$$(\Xi \circ \Phi^{eq})\,(\omega^* E_{\zeta,f}) \cong U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \tag{5.4}$$

of vector spaces. Hence we have only to show the following.

Lemma 5.6. *Under the identification* (5.4) *the left $C_\zeta^{\leqq 0}$-comodule structure of $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ is associated to the right $U_\zeta^{L,\leqq 0}$-module structure given by*

$$(u \otimes e(\lambda)) \cdot v = \mathrm{ad}(Sv)(u) \otimes e(\lambda) \qquad (u \in U_{\zeta,f}, \lambda \in \Lambda, v \in U_\zeta^{L,\leqq 0}).$$

Proof. Note that the left $C_\zeta^{\leqq 0}$-comodule structure of $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ is given by

$$U_{\zeta,f} \otimes \mathbb{C}[\Lambda] \cong \Xi(C_\zeta \otimes (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])),$$

where $C_\zeta \otimes (U_{\zeta,f} \otimes \mathbb{C}[\Lambda])$ is regarded as a left $C_\zeta^{\leqq 0}$-comodule by the tensor product of C_ζ (with left $C_\zeta^{\leqq 0}$-comodule structure $(\mathrm{res} \otimes 1) \circ \Delta : C_\zeta \to C_\zeta^{\leqq 0} \otimes C_\zeta$) and $U_{\zeta,f} \otimes \mathbb{C}[\Lambda]$ with trivial left $C_\zeta^{\leqq 0}$-comodule structure. Hence it is sufficient to show that for a right C_ζ-comodule M the right $U_\zeta^{L,\leqq 0}$-module structure of

$$M \cong \Xi(C_\zeta \otimes M) \in \mathrm{Comod}(C_\zeta^{\leqq 0})$$

is given by

$$m \cdot v = (Sv) \cdot m \quad (m \in M, v \in U_\zeta^{L,\leqq 0}).$$

Denote by M^{triv} the trivial right C_ζ-comodule which coincides with M as a vector space. We denote by $M \ni m \leftrightarrow \overline{m} \in M^{triv}$ the canonical

linear isomorphism. We have $C_\zeta \otimes M^{triv} \in \mathrm{Comod}^r(C_\zeta)$ as the tensor product of $C_\zeta \in \mathrm{Comod}^r(C_\zeta)$ and $M^{triv} \in \mathrm{Comod}^r(C_\zeta)$. We can also define a left $C_\zeta^{\leqq 0}$-comodule structure of $C_\zeta \otimes M^{triv}$ as the tensor product of the left $C_\zeta^{\leqq 0}$-comodules C_ζ and M^{triv} where the left $C_\zeta^{\leqq 0}$-comodule structure of M^{triv} is given by the right $U_\zeta^{L,\leqq 0}$-module structure

$$\overline{m} \cdot v = \overline{(Sv) \cdot m} \qquad (m \in M, v \in U_\zeta^{L,\leqq 0}).$$

Then we have an linear isomorphism

$$C_\zeta \otimes M \ni \varphi \otimes m \mapsto \sum_{(m)} \varphi m_{(1)} \otimes \overline{m_{(0)}} \in C_\zeta \otimes M^{triv}$$

preserving the right C_ζ-comodule structures and the left $C_\zeta^{\leqq 0}$-comodule structures. It follows that

$$\Xi(C_\zeta \otimes M) \cong \Xi(C_\zeta \otimes M^{triv}) = M^{triv} \in \mathrm{Comod}(C_\zeta^{\leqq 0}).$$

□

The proof of Proposition 5.1 is complete.

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